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An output expression for a dispersive delay pulse compression filter under arbitrary inputs.

Jones, William Robert

Monterey, California. U.S. Naval Postgraduate School

AN OUTPUT EXPRESSION FOR A DISPERSIVE DELAY PULSE COMPRESSION FILTER UNDER ARBITRARY INPUTS

WILLIAM ROBERT JONES

AN OUTPUT EXPRESSION FOR A DISPERSIVE DELAY PULSE

COMPRESSION FILTER UNDER ARBITRARY INPUTS

by

William Robert Jones
Lieutenant, United States Navy
B.A., Rice Institute, 1959

[redacted]

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ABSTRACT

Pulse compression filters are used extensively in modern radar systems. The nature of output waveforms from dispersive delay pulse compression filters driven by specific matched input waveforms has been studied in great detail for these radar applications. However, little work has been done to generalize these results. This paper obtains an expression for the filter output in terms of arbitrary input signals. Several particular input waveforms are analyzed using an ideal filter with assumed specific characteristics. In an attempt to indicate trends, different pulse widths and linear frequency modulation rates are assumed for the pulse shapes chosen. The resulting output envelopes are plotted graphically.

The author wishes to express his appreciation for the assistance and encouragement given him by Professor Glen A. Myers of the U. S. Naval Postgraduate School.

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1. Introduction.

The development of radar, starting prior to World War II, and continuing to the present, has fostered many concepts in the areas of data processing and signal design. Early systems used single R-F pulses for echo ranging. Range was necessarily limited by average power transmitted, necessitating long time duration pulses for long detection ranges. Unfortunately, range resolution decreases as pulse width increases. Initial attempts to resolve this conflict were merely extensions of the basic single pulse approach: Increase peak power, thereby increasing average transmitted power and hence range, without increasing pulse width or degrading range resolution.

The limitations of this approach were obvious. Thus much effort was, and still is, being directed toward a better understanding of the basic factors leading to optimum performance of radar systems. This led to the concept of matched filtering. This involves the "design" of an input signal which is constrained or "matched" to the receiver system so as to minimize apparent pulse width, maintain average signal power and maximize signal to noise ratio. The technique of designing an echo ranging system employing a long time duration, frequency modulated signal, and a matched receiver to "compress" this signal is the chirp concept. This concept was arrived at independently by several men, among them Dicke [1] , and Darlington [2]. Their work, conducted independently in the early 1950's, represented an attempt to

define theoretical performance limitations in pulse radar systems. This led to U.S. Patents being issued to the two men in January, 1953 and May, 1954 respectively. Their theoretical work was extensively developed and given practical application by Klauder [3] at Bell Telephone Laboratory and Cook [4] at Sperry Gyroscope Company, among others.

That this is one practical approach to the radar problem is evidenced by the many operational systems in existance today which utilize its concept [5]. However, this general concept has benefits throughout the fields of radio communications and data processing.

The nature of output waveforms obtained from these dispersive delay lines driven by specific matched input waveforms has been studied in detail for radar applications. Little work, however, has been done to generalize these results. This paper obtains an expression for output waveforms from the filter in terms of a general input signal. Thus, while it is not the intent here to present any new uses for the chirp filter or dispersive delay line, the nature of this analysis should be useful to those concerned with such applications.

The first portion of this paper is devoted to the derivation of a general expression for the waveform out of a pulse compression filter in terms of an arbitrary time envelope input. This results in a formal integral equation, equation (6).

As examples, several inputs, some matched to the filter and others mismatched, are then examined in some detail, using an ideal filter with specified characteristics. The mathmatical functions derived for output envelopes of these examples are plotted as figures (4) through (17). In an attempt to indicate trends and allow interpolation between curves, several different pulse widths and linear frequency modulation rates are chosen.

No attempt is made to examine output phase characteristics in detail. However the analytical expressions for phase, as a function of time, are derived for completeness.

Several general observations and interpretations of output characteristics are offered in the Conclusions and Interpretations section.

Derivations for output expressions for the specific input examples chosen are included as Appendices I and II.

2. Analysis and Results.

An expression for the output waveform of the dispersive filter in terms of the input time function and the filter characteristic will now be derived. Figure (1) illustrates the problem in simple form.

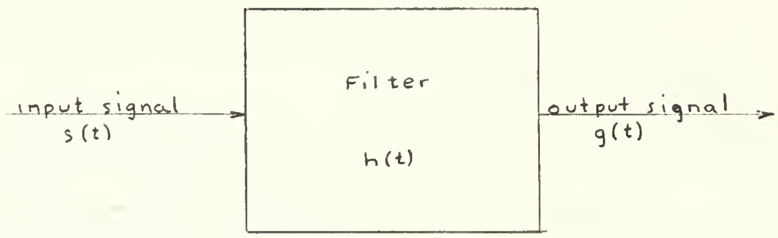

Figure 1. Indication of the parameters of interest.

Given an input function s(t), passing through a linear filter having impulse response h(t), it is desired to obtain an expression for the output, g(t). Since the filter is a linear device, this can be expressed as

$$G(\omega) = S(\omega)\,H(\omega) \tag{1}$$

where $G(\omega)$ and $S(\omega)$ are the Fourier transforms of the output and input time functions respectively, and $H(\omega)$ is the filter transfer function. That is

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega \tag{2}$$

and

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, d\omega$$

Then from (1) and (2)

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, H(\omega) e^{j\omega t}\, d\omega \qquad (3)$$

As noted in the introduction, expressions for g(t) have been obtained for particular input functions by Klauder [3], Cook [4], and Fowle [6]. An expression for the output for an arbitrary input signal is desired.

In general, the transfer function of a linear filter is complex and can therefore be written

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

where $|H(\omega)|$ is the magnitude and $\theta(\omega)$ is the phase of $H(\omega)$. For a dispersive filter

$$\theta(\omega) = \frac{(\omega - \omega_0)^2}{2k}$$

where ω_0 is the center frequency at which no phase shift (no time delay) is introduced by the filter, and k is the rate of change of time delay with frequency. The time delay is then given by

$$\tau(\omega) = -\frac{d\,\theta(\omega)}{d\omega} = \frac{\omega - \omega_0}{k}$$

This relation is shown graphically in figure (2).

The input time function is defined as

$$s(t) = f(t)\, e^{j\phi(t)}$$

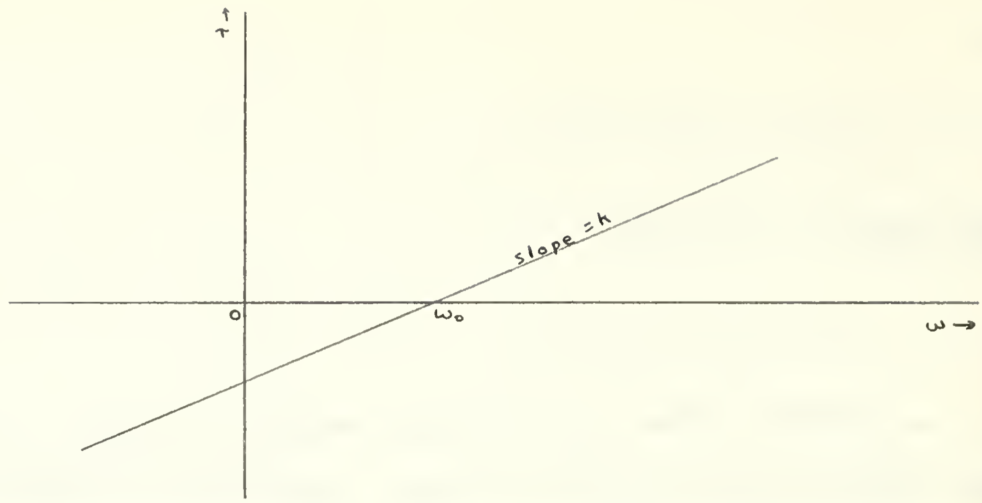

Figure 2. Plot of time delay τ , versus input frequency ω.

where $\phi(t)$ is the time phase and f(t) is an arbitrary time function. It will simplify the mathematics if, at this point, $\phi(t)$ is defined as

$$\phi(t) = \omega_1 t + \frac{k}{2} t^2$$

The instantaneous frequency ω_i is then $\frac{d\phi}{dt}$ or

$$\omega_i = \omega_1 + kt$$

From (3), the output time function is

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \exp\left[j \frac{(\omega - \omega_0)^2}{2k}\right] \exp[j\omega t] \, d\omega$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\frac{(\omega-\omega_0)^2}{2k}} e^{j\omega t} \int_{-\infty}^{\infty} f(\tau) e^{j(\omega_1 \tau + \frac{k}{2}\tau^2 - \omega\tau)} \, d\tau \, d\omega \qquad (4)$$

where τ is a variable of integration. (It should be pointed out that g(t) will, in general, be a complex function. The physical output time function is obtained by taking the real part of g(t).)

Exchanging the order of integration yields

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau)\, e^{j\left(\omega_1 \tau + \frac{k}{2}\tau^2 + \frac{\omega_0^2}{2k}\right)} \int_{-\infty}^{\infty} e^{j\left[\frac{\omega^2}{2k} - \frac{\omega\omega_0}{k} + (t-\tau)\omega\right]}\, d\omega\, d\tau$$

which can be expressed as

$$g(t) = \frac{1}{2\pi}\, e^{j\frac{\omega_0^2}{2k}} \int_{-\infty}^{\infty} f(\tau)\, e^{j\left[\omega_1\tau + \frac{k}{2}\tau^2 - \frac{(\omega_0 + k\tau - kt)^2}{2k}\right]} \int_{-\infty}^{\infty} e^{\frac{j}{2k}\left[\omega - \omega_0 + k(t-\tau)\right]^2}\, d\omega\, d\tau$$

Letting $\sqrt{2k}\,u = \omega - \omega_0 + k(t-\tau)$ and performing necessary algebraic manipulation gives

$$g(t) = \frac{\sqrt{2k}}{2\pi}\, e^{j\left(\omega_0 t - \frac{k}{2}t^2\right)} \int_{-\infty}^{\infty} f(\tau)\, e^{j(\omega_1 - \omega_0 + kt)\tau} \int_{-\infty}^{\infty} e^{ju^2}\, du\, d\tau$$

Recognizing that

$$\int_{-\infty}^{\infty} e^{ju^2}\, du = \sqrt{\pi}\, e^{j\frac{\pi}{4}} \tag{5}$$

the output becomes

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j\left(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4}\right)} \int_{-\infty}^{\infty} f(\tau)\, e^{j(\omega_1 - \omega_0 + kt)\tau}\, d\tau$$

Realizing that the term $(\omega_1 - \omega_0)$ represents merely a constant phase shift through the filter, ω_1 may now be set equal to ω_0 with no loss of generality.

Carrying out this substitution gives

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j\left(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4}\right)} \int_{-\infty}^{\infty} f(\tau)\, e^{jkt\tau}\, d\tau \tag{6}$$

By defining $F(\omega)$ as the Fourier transform of $f(t)$, the output may be written

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j\left(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4}\right)}\, F(-kt) \tag{7}$$

This result shows that for an input signal which consists of an arbitrary envelope, f(t), and a carrier which is being frequency modulated in a linear fashion, i.e.

$$\phi(t) = \omega_0 t + \frac{k}{2} t^2$$

the output of the filter consists of an envelope whose time function may be thought of as the Fourier transform of the input, with variable (-kt) replacing the more familiar (ω), and a carrier which is being frequency modulated at the same rate as the input, but in the opposite sense, i.e.

$$\phi_0(t) = \omega_0 t - \frac{k}{2} t^2$$

In particular, two cases of interest have been rather extensively investigated: (1) When the input envelope is a rectangular pulse of duration 2T [7], and (2) When the input envelope is gaussian [6], i.e.

$$f(t) = e^{-a^2 t^2}$$

This form of the output, equation (6), can be checked by using results obtained previously by others.

Applying (6) to case (1) gives

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-\infty}^{\infty} f(\tau)\, e^{jkt\tau}\, d\tau$$

Substituting

$$f(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

gives

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-T}^{T} e^{jkt\tau}\, d\tau \tag{8}$$

which when integrated yields

$$g(t) = \sqrt{\frac{2k}{\pi}}\, T\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \; \frac{\sin kTt}{kTt} \tag{9}$$

and agrees with results of Chin [7].

For case (2)

$$f(t) = e^{-a^2 t^2}$$

Again applying equation (6)

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-\infty}^{\infty} e^{-a^2\tau^2} e^{jkt\tau}\, d\tau$$

which when integrated gives

$$g(t) = \sqrt{\frac{k}{2a^2}}\, e^{-(\frac{k}{2a})^2 t^2}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \tag{10}$$

It is seen that the output envelope is also gaussian which agrees with Fowle, et.al. [6].

Another waveform is now considered to indicate the general nature of the derived expression. Let

$$f(t) = \begin{cases} \cos\omega_1 t & -\frac{\pi}{2\omega_1} \le t \le \frac{\pi}{2\omega_1} \\ 0 & \text{elsewhere} \end{cases} \qquad \omega_1 \ll \omega_0$$

then from (6)

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-\frac{\pi}{2\omega_1}}^{\frac{\pi}{2\omega_1}} \cos\omega_1\tau\, e^{jkt\tau}\, d\tau$$

which after integration and suitable algebraic manipulation may be expressed as

$$g(t) = \sqrt{\frac{2k}{\pi}} \; \frac{\omega_1}{\omega_1^2 - (kt)^2} \; \cos \frac{\pi k t}{2\omega_1} \; e^{j\left(\omega_0 t - \frac{k}{2} t^2 + \frac{\pi}{4}\right)} \tag{11}$$

Figure (3) shows the three output envelopes plotted as time functions with pulse width, T, and filter constant, k, as parameters. Figures (4), (6), and (8) respectively show the three cases for varying pulse widths and a particular value of filter constant.

Mismatched Input Signals

The previous development has considered cases where the input signal was matched to the filter. That is, the carrier was being linearly frequency modulated at a rate equal to the filter constant.[1] If the derived output expression is to have general application, it should be useful also for input signals which are not matched to the filter. The input for the preceding analysis was specified as

$$s(t) = f(t)\, e^{j\left(\omega_0 t + \frac{k}{2} t^2\right)}$$

For inputs of the form

$$s(t) = x(t)\, e^{j(\omega_0 t)}$$

[1]A filter is said to be matched to a particular signal if the filter transfer function is the complex conjugate of the signal. The dispersive filter with transfer function $H(\omega) = e^{j\frac{(\omega-\omega_0)^2}{2k}}$ is thus matched to an input of the form $s(t) = f(t)\, e^{j(\omega_0 t + \frac{k}{2} t^2)}$, as shown by Chin and Cook [7], where $f(t) = 1 \;\; -T \le t \le T$ and 0 elsewhere.

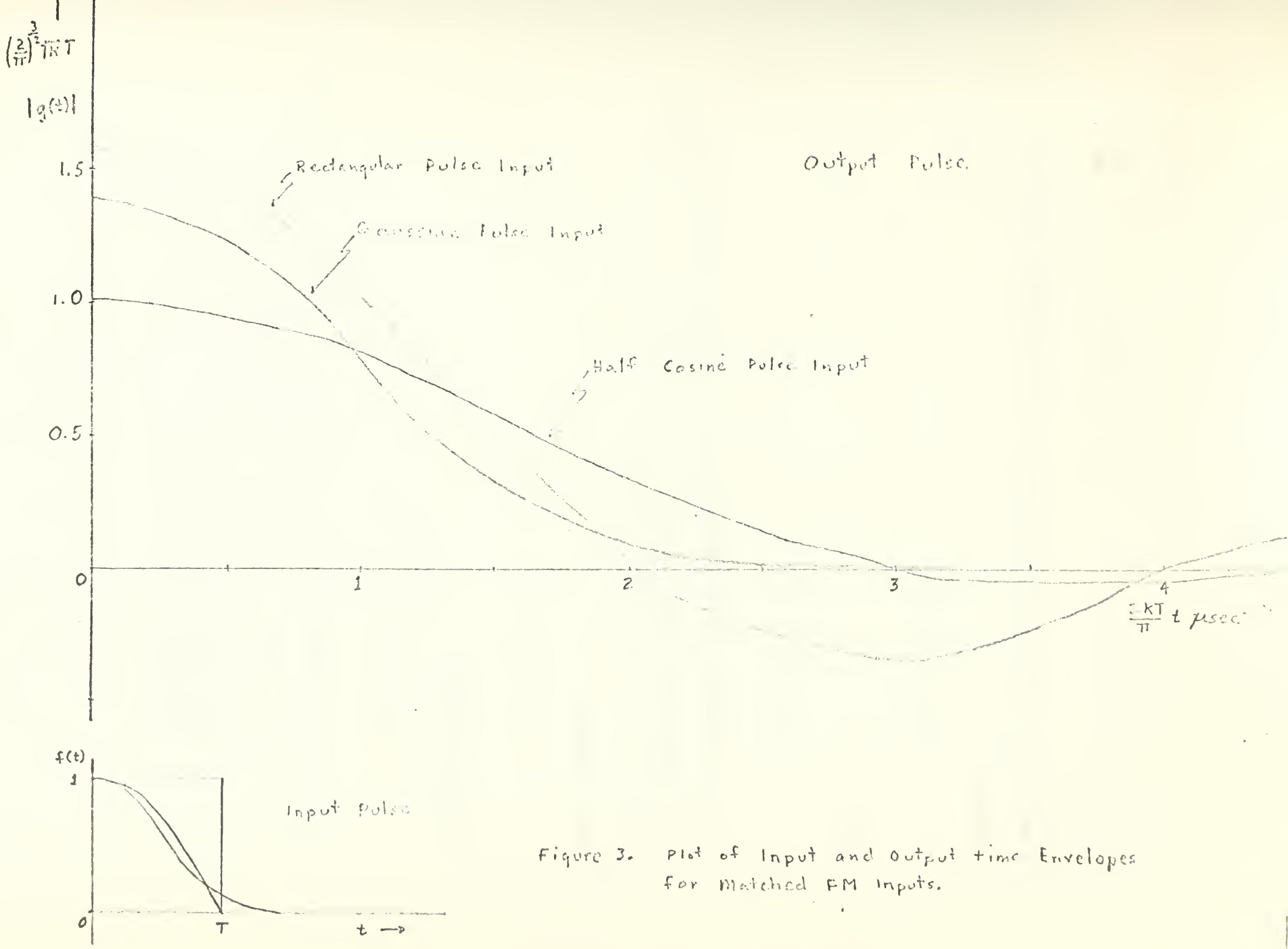

Figure 3. Plot of Input and Output time Envelopes for Matched FM Inputs.

or

$$s(t) = y(t)\, e^{j(\omega_o t + \frac{\mu}{2} t^2)} \qquad \mu \neq k$$

then f(t), which is an arbitrary time function, can be defined as

$$f(t) = x(t)\, e^{-j\frac{k}{2}t^2}$$

Then

$$s(t) = x(t) e^{j\omega_o t}$$

and the expression obtained in (6) is thus applicable to cases where the input signal is not matched to the filter. Similarly, for different modulation rates, f(t) may be expressed as

$$f(t) = y(t)\, e^{j\frac{(\mu - k)}{2}t^2}$$

giving an input

$$s(t) = y(t)\, e^{j(\omega_o t + \frac{\mu}{2} t^2)}$$

The problem becomes one of evaluating the right side of (6). Fortunately, there is a large class of useful functions where this evaluation is not only possible, but quite direct. The three envelope shapes previously considered belong to this class.

Constant Frequency Carrier Inputs.

The case where the input signal is a constant frequency carrier having a rectangular pulse envelope will now be

considered. Let

$$f(t) = x(t)\, e^{-j\frac{k}{2}t^2}$$

$$x(t) = \begin{cases} 1 & -T \le t \le T \\ 0 & \text{elsewhere} \end{cases}$$

Then

$$s(t) = x(t)\, e^{j\omega_0 t}$$

Applying (6)

$$g(t) = \sqrt{\frac{k}{2\pi}}\; e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-T}^{T} e^{-j\left(\frac{k}{2}\tau^2 - kt\tau\right)}\, d\tau \qquad (12)$$

Completing the square in the exponential

$$g(t) = \sqrt{\frac{k}{2\pi}}\; e^{j(\omega_0 t + \frac{\pi}{4})} \int_{-T}^{T} e^{-j\frac{k}{2}(\tau - t)^2}\, d\tau$$

Then substituting $\sqrt{\frac{\pi}{2}}\, u = \sqrt{\frac{k}{2}}(\tau - t)$

$$g(t) = \frac{1}{\sqrt{2}}\, e^{j(\omega_0 t + \frac{\pi}{4})} \int_{-\sqrt{k/\pi}\,(T+t)}^{\sqrt{k/\pi}\,(T-t)} e^{-ju^2}\, du \qquad (13)$$

The real part of the output is expressed as an envelope

$$En\{g(t)\} = \frac{1}{\sqrt{2}}\, C\left[-\sqrt{\tfrac{k}{\pi}}(t+T)\right] + \frac{1}{\sqrt{2}}\, C\left[\sqrt{\tfrac{k}{\pi}}(T-t)\right] \qquad (14)$$

and a phase term

$$\phi(t) = \omega_0 t + \frac{\pi}{4} + \tan^{-1}\left[\frac{-C\left[\sqrt{\frac{k}{\pi}}(T+t)\right] + C\left[\sqrt{\frac{k}{\pi}}(T-t)\right]}{S\left[\sqrt{\frac{k}{\pi}}(T+t)\right] + S\left[\sqrt{\frac{k}{\pi}}(T-t)\right]}\right] \qquad (15)$$

where C(x) and S(x) are the Fresnel cosine and sine integrals respectively defined as

$$C(x) = \int_0^x \cos \frac{\pi}{2} t^2 \, dt$$

$$S(x) = \int_0^x \sin \frac{\pi}{2} t^2 \, dt$$

A plot of the real envelope of g(t) for varying pulse widths is shown in figure (5).

If the input envelope is gaussian rather than rectangular, and the above analysis is applied, the resulting output envelope is also gaussian and of the form

$$g(t) = \sqrt{\frac{k}{2}} \left[a^4 + \left(\frac{k}{2}\right)^2\right]^{-\frac{1}{4}} e^{-\left[\frac{\left(\frac{a}{2}k\right)^2 t^2}{a^4 + \left(\frac{k}{2}\right)^2}\right]} e^{j\left[\omega_0 t - \left(1 + \frac{k^2}{4a^2 + k^2}\right)\frac{k}{2} t^2 + \frac{\pi}{4} + \tan^{-1}\frac{k}{2a^2}\right]} \tag{16}$$

Again, taking the real part of the envelope and phase yields

$$En\{g(t)\} = \sqrt{\frac{k}{2}} \left(a^4 + \frac{k^2}{4}\right)^{-\frac{1}{4}} e^{-\left[\frac{(ak)^2 t^2}{4a^4 + k^2}\right]} \tag{17}$$

and

$$\phi(t) = \omega_0 t - \left(1 + \frac{k^2}{4a^2 + k^2}\right)\frac{k}{2} t^2 + \frac{\pi}{4} + \tan^{-1}\left(\frac{k}{2a^2}\right) \tag{18}$$

For the case of a half cosine wave envelope

$$s(t) = \begin{cases} \cos \omega_1 t \, e^{j\omega_0 t} & -\frac{\pi}{2\omega_1} \le t \le \frac{\pi}{2\omega_1} \\ 0 & \text{elsewhere} \end{cases} \qquad \omega_1 \ll \omega_0$$

the output becomes

$$g(t) = \frac{1}{2\sqrt{2}} e^{j\left(\omega_0 t + \frac{\pi}{4} + \frac{\omega_1^2}{2k}\right)} \left[e^{j\omega_1 t} \int_a^b e^{-j\frac{\pi}{2}u^2} du + e^{-j\omega_1 t} \int_c^d e^{-j\frac{\pi}{2}v^2} dv\right] \tag{19}$$

where

$$a = \sqrt{\frac{k}{\pi}}\left[t + \frac{\omega_1}{k} - \frac{\pi}{2\omega_1}\right] \qquad c = \sqrt{\frac{k}{\pi}}\left[t - \frac{\omega_1}{k} - \frac{\pi}{2\omega_1}\right]$$

$$b = \sqrt{\frac{k}{\pi}}\left[t + \frac{\omega_1}{k} + \frac{\pi}{2\omega_1}\right] \qquad d = \sqrt{\frac{k}{\pi}}\left[t - \frac{\omega_1}{k} + \frac{\pi}{2\omega_1}\right]$$

Derivations of the above expressions are shown in Appendix I. The resulting outputs are plotted in figures (7) and (9) respectively.

Mismatched Linear FM Inputs

The two special input functions, constant carrier frequency and matched linear frequency modulated carrier have been examined. It is now appropriate to look at inputs in which the carrier is linearly frequency modulated at a rate different from the filter constant. The filter transfer function remains

$$H(\omega) = e^{j\frac{(\omega-\omega_0)^2}{2k}}$$

But the input is now defined as

$$s(t) = y(t)\, e^{j(\omega_0 t + \frac{\mu}{2}t^2)}$$

where y(t) is an arbitrary time function. The instantaneous carrier frequency for s(t) is then

$$\omega_i = \omega_0 + \mu t$$

This type of input can be separated into two classes; those inputs for which μ is less than k, and those for which μ is greater than k. A detailed analysis of both cases is shown in Appendix II where y(t) is defined as

$$y(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

The results are included here to indicate filter behavior.

For the case μ less than k, the output is

$$g(t) = \left(\frac{k}{k-\mu}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}}\, e^{j\left(\omega_0 t - \left(1 - \frac{k}{k-\mu}\right)\frac{k}{2}t^2 + \frac{\pi}{4}\right)} \int_a^b e^{-j\frac{\pi}{2}u^2}\, du \qquad \mu < k \tag{20}$$

where

$$a = -\left(\frac{k-\mu}{\pi}\right)^{\frac{1}{2}}\left[T + \frac{kt}{k-\mu}\right] \qquad\qquad b = \left(\frac{k-\mu}{\pi}\right)^{\frac{1}{2}}\left[T - \frac{kt}{k-\mu}\right]$$

giving an envelope

$$E_n\{g(t)\} = \left[\frac{k}{k-\mu}\right]^{\frac{1}{2}} \frac{1}{\sqrt{2}} \left[\, C(b) - C(a)\right] \tag{21}$$

and phase

$$\phi(t) = \omega_0 t - \left(1 - \frac{k}{k-\mu}\right)\frac{k}{2}t^2 + \frac{\pi}{4} + \tan^{-1}\left[-\frac{C(b) - C(a)}{S(b) - S(a)}\right] \tag{22}$$

Figures (10) through (17) show output envelopes for varying pulse widths and values of μ between k and -k. If μ is set equal to zero, the resulting expression for g(t) should agree with equation (13) (constant frequency rectangular envelope input). Carrying out this substitution, the output reduces to

$$g(t) = \frac{1}{\sqrt{2}}\, e^{j(\omega_0 t + \frac{\pi}{4})} \int_{a'}^{b'} e^{-j\frac{\pi}{2}u^2}\, du$$

where

$$a' = -\sqrt{\frac{k}{\pi}}\,(T+t) \qquad\qquad b = \sqrt{\frac{k}{\pi}}\,(T-t)$$

which shows the analysis is consistent at this point. For the same function y(t) but for μ greater than k, the output becomes

$$g(t) = \frac{1}{\sqrt{2}}\left[\frac{k}{\mu-k}\right]^{\frac{1}{2}} e^{j\left(\omega_0 t + \frac{\pi}{4} - \left(\frac{k}{\mu-k}+1\right)\frac{k}{2}t^2\right)} \int_a^b e^{j\frac{\pi}{2}v^2}\, dv \qquad \mu > k \tag{23}$$

where

$$a = -\left(\frac{\mu-\kappa}{\pi}\right)^{\frac{1}{2}}\left[T+\frac{\kappa t}{\mu-\kappa}\right] \qquad\qquad b = \left(\frac{\mu-\kappa}{\pi}\right)^{\frac{1}{2}}\left[T-\frac{\kappa t}{\mu-\kappa}\right]$$

A somewhat surprising result is that the output envelope depends only on $|k-\mu|$ and not on the sign of this difference. I.e.

$$E_n\{q(t)\} = \frac{1}{\sqrt{2}}\left[\frac{\kappa}{|\kappa-\mu|}\right]^{\frac{1}{2}}\left[C\left[\left(\frac{|\mu-\kappa|}{\pi}\right)^{\frac{1}{2}}\left(T-\frac{\kappa t}{|\kappa-\mu|}\right)\right] - C\left[\left(\frac{|\mu-\kappa|}{\pi}\right)^{\frac{1}{2}}\left(T+\frac{\kappa T}{|\kappa-\mu|}\right)\right]\right] \qquad (24)$$

where C(x) is the Fresnel cosine integral as previously defined. The difference in the outputs for the two cases lies only in the phase of the carrier.

3. Conclusions and Intrepretations.

Thus far, this paper has developed mathematical expressions for the chirp filter output waveforms. These expressions are useful to the extent that they permit interpretations of the outputs in terms of physical applications. Some of the properties of the output waveforms, available through examination of the expressions and their associated graphs, are presented here.

Figures (5a), (7a) and (9a) show output envelopes for relatively wide input envelope of the three shapes previously considered. The carrier frequency of these envelopes is held constant throughout the pulse, i.e.; the input signal is purposely mismatched to the filter. The characteristics of the filter are such that if a pure sinusoid is used to drive the filter, the only difference between input and output waveforms is a phase shift proportional to the frequency of the sinusoid. For a constant carrier frequency pulse of long duration, the major contribution to the frequency spectrum is the carrier frequency itself with only minor portions due to the amplitude modulation of the pulse envelope, independent of its shape.[1] Thus for long pulses it is reasonable to expect that the output waveform from the filter should approach the input. In the limit as T

[1]The principle of time-bandwidth product invariance would indicate that, as time duration of pulse increases, energy content in the frequency spectrum becomes concentrated in a narrowing frequency region about the carrier frequency.

approaches ∞ the two are identical and filter is just a delay line. Referring to figures (5a), (7a), and (9a), this appears to be true. The outputs for 100 microsecond half pulse width inputs bear a greater resemblance to these inputs than the 10 microsecond or even 50 microsecond half pulse widths. However, as expected, this difference is not as noticeable in the gaussian case, where amplitude modulation effects are minimized (a characteristic of the gaussian envelope).

Figures (4d) through (9d) show the output envelopes for very narrow input pulse widths. In this case the major frequency components in the waveform are due to amplitude modulations rather than carrier frequency variations. For very narrow pulses with similar maximum amplitude, the frequency spectrum, to a rough approximation, changes only slightly with pulse shape. This being the case, the output pulse shape should remain approximately the same for all narrow input pulse shapes providing the energy content is of the same order of magnitude, and maximum amplitudes and pulse widths change but little. Also, when the pulse is narrow, the carrier frequency changes within the pulse are small, even when the carrier is being linearly frequency modulated. Thus, whether the input signal is matched or mismatched to the filter, the slow variations in the output will be only slightly affected. Comparing figure (4d) with (5d) and figure (8d) with (9d) respectively, it is noted that the fine structure apparent in the output for the

mismatched case disappears in the matched case, yielding a sort of "envelope of the envelope" output. The actual reason for this phenomenon escapes the rather simple mathematical derivations presented here. However, the more detailed analysis of the rectangular pulse, which includes intermediate stages between constant carrier frequency pulses and matched frequency modulated carrier pulses, indicates a possible explanation. As the input approaches a matched input, the output envelope oscillations become so rapid they disappear altogether, leaving only the slowly varying waveform shown in figure (4d). Due to characteristics inherent in the gaussian envelope, e.g. having a finite value for all time, it does not exhibit this property which is apparent in the time limited envelopes. However, the output envelope for a narrow gaussian envelope input certainly resembles closely the slowly varying portion of the output for the rectangular pulse and the half cosine pulses.

The output for a gaussian input envelope also exhibits other interesting properties. As shown in equations (11) and (16) and in figures (5) and (7), the output envelope, when the filter is driven by a waveform with a gaussian envelope, is always gaussian. This is true whether the carrier frequency is constant or is being linearly frequency modulated. Though it does not appear intuitively obvious that all the frequency components will be combined within the filter to yield this result, it can be predicted once equation (6) is obtained. One interesting property of a gaussian

pulse is that its Fourier transform is also a gaussian pulse. Equation (6) indicates that the chirp filter can be regarded as giving an output which is the Fourier transform of the input. Thus for a gaussian envelope input, it would be expected that the output envelope would be gaussian.

During this study two points which might be of particular interest at the present time were uncovered: an insight into the effect of doppler shift in returning radar signals on receiver output; and the possible use of a dispersive delay line of the chirp filter type as a spectrum analyzer.

It is evident in examining figures (10) thru (17) and comparing equation (9) with equation (10) that mismatched signal inputs will affect the output signal. In particular, if the frequency modulation rate of the input signal is not exactly matched to the filter itself, the output envelope assumes a different shape. In papers concerned with radar utilization of the chirp concept, the effect of target velocity, or doppler frequency shift, on incoming signals, has been treated as if the change in modulation rate of the return echo is not present. That is, over the frequency interval of interest, doppler shift is considered a function only of target velocity, independent of frequency. However, the magnitude of doppler shift is actually proportional not only to velocity, but to frequency. Thus the effect of a moving target on a chirp type radar would be to change the modulation rate of the input. Therefore the output would not only be retarded or advanced in time, but

the actual output envelope would assume a different shape. Perhaps this envelope shape change could be detected and used to correct the range error introduced by the time delay. It should be noted however, that the envelope shape by itself will not indicate direction of doppler shift but only magnitude. This is seen in equation (24) which shows that the envelope is a function of the **absolute** value of the difference between the filter constant and the signal frequency modulation rate and does not depend on the sign of this difference.

As indicated previously, the chirp filter may be thought of as giving an output waveform, in time, which under suitable carrier modulation, is the Fourier transform of the input time envelope waveform, i.e.

$$q(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-\infty}^{\infty} f(\tau)\, e^{jkt\tau}\, d\tau$$

This gives rise to the concept of a spectrum analyzer which consists of an input section and a chirp type filter. The input section generates a carrier which is being frequency modulated at the filter constant rate and which could be amplitude modulated by the input signal being analyzed. By suitable calibration of an output time display, the actual frequency components of the input signal could be shown. [16]

Figure (18a) shows output pulse width as a function of input pulse width for a chirping rectangular pulse envelope. It is interesting to note that the product of output pulse

width times input pulse width is a constant. I.e. if input pulse width is 2T and output envelope varies as $\frac{\sin KTt}{KTt}$ then output pulse width times input pulse width equals $2T\left(\frac{\pi}{2KT}\right)$ or $\frac{\pi}{K}$. Thus for input pulses longer than $\frac{\pi}{K}$, the filter acts as a pulse compressor, whereas for inputs of shorter duration than $\frac{\pi}{K}$, it is effectively a pulse stretcher.

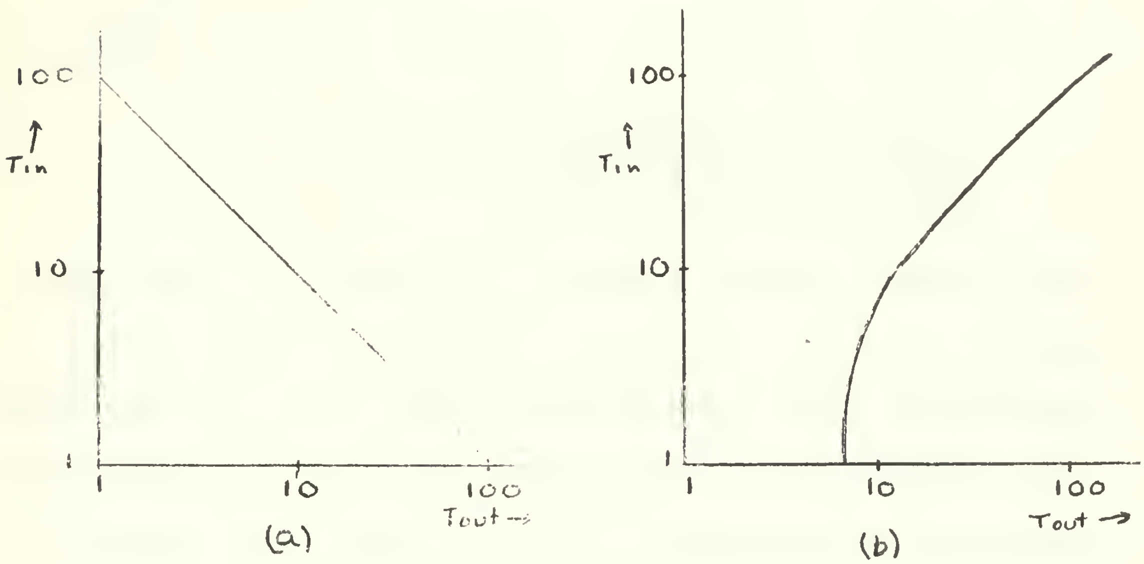

Figure 18. Plot of output pulse width versus input pulse width for rectangular pulse envelope. (a) matched frequency modulated carrier input. (b) constant frequency carrier input.

This property is also exhibited for the gaussian case as shown in figure (19a). Consider the pulse width for a gaussian pulse to be the time between points which are e^{-1} of the maximum value. If the input is given by $s(t) = e^{-a^2t^2}$ then from (11), the output varies as $e^{-\left(\frac{K}{2a}\right)^2 t^2}$ and the "pulse widths" are $\tau = \pm\frac{1}{a}$ and $\tau = \pm\frac{2a}{K}$ respectively. The product of input and output pulse widths thus defined is equal to $\frac{2}{K}$.

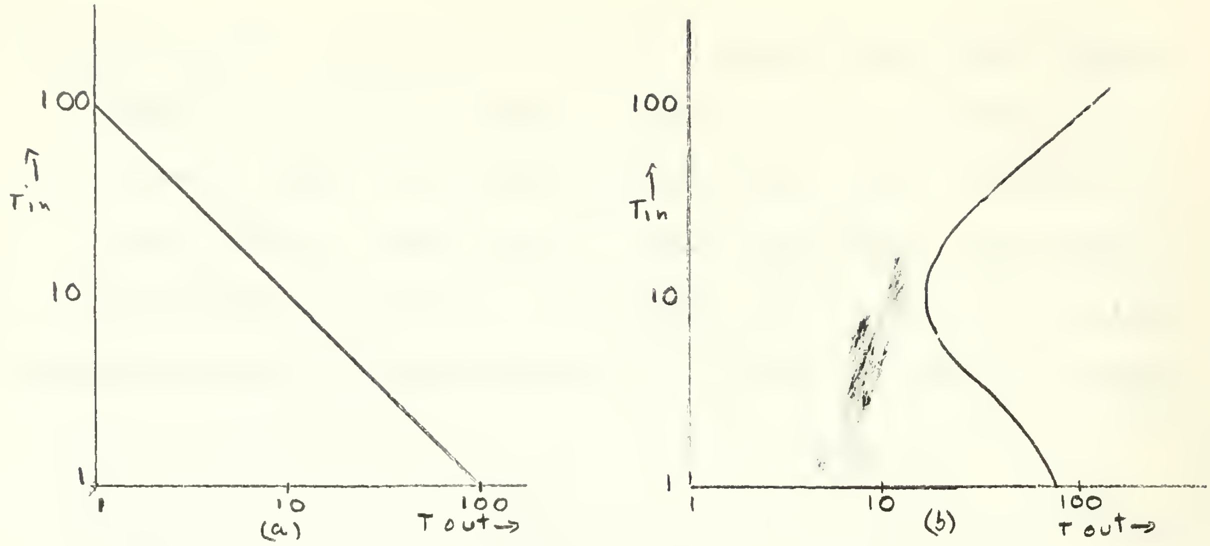

Figure 19. Plot of output e^{-1} points versus input e^{-1} points, gaussian envelope. (a) matched frequency modulated input, (b) constant frequency carrier.

For the half cosine envelope the product of first zero crossing of input and output for the matched case is also a constant, $\frac{3}{2}\frac{\pi}{k}$. This relationship is plotted in figure (20a).

The constant carrier frequency pulse widths are plotted in each case for comparison and to indicate how compression effect is lost when input signal is mismatched.

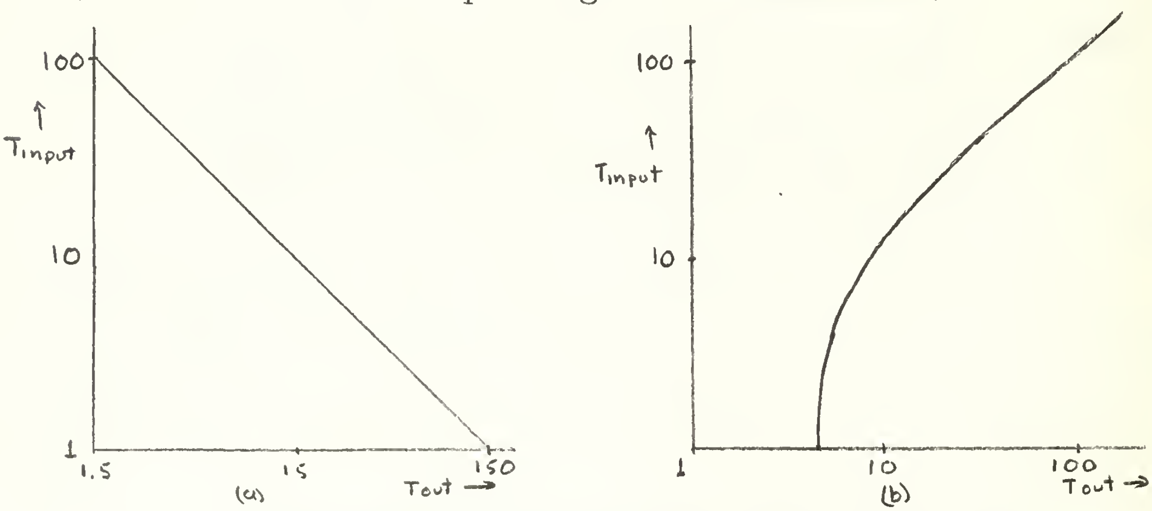

Figure 20. Plot of first zero crossing, input versus output, half cosine envelope. (a) matched frequency modulated carrier, and (b) constant frequency carrier.

Finally, figure (21) shows output pulse width as a function of input carrier modulation rate for constant input pulse widths.

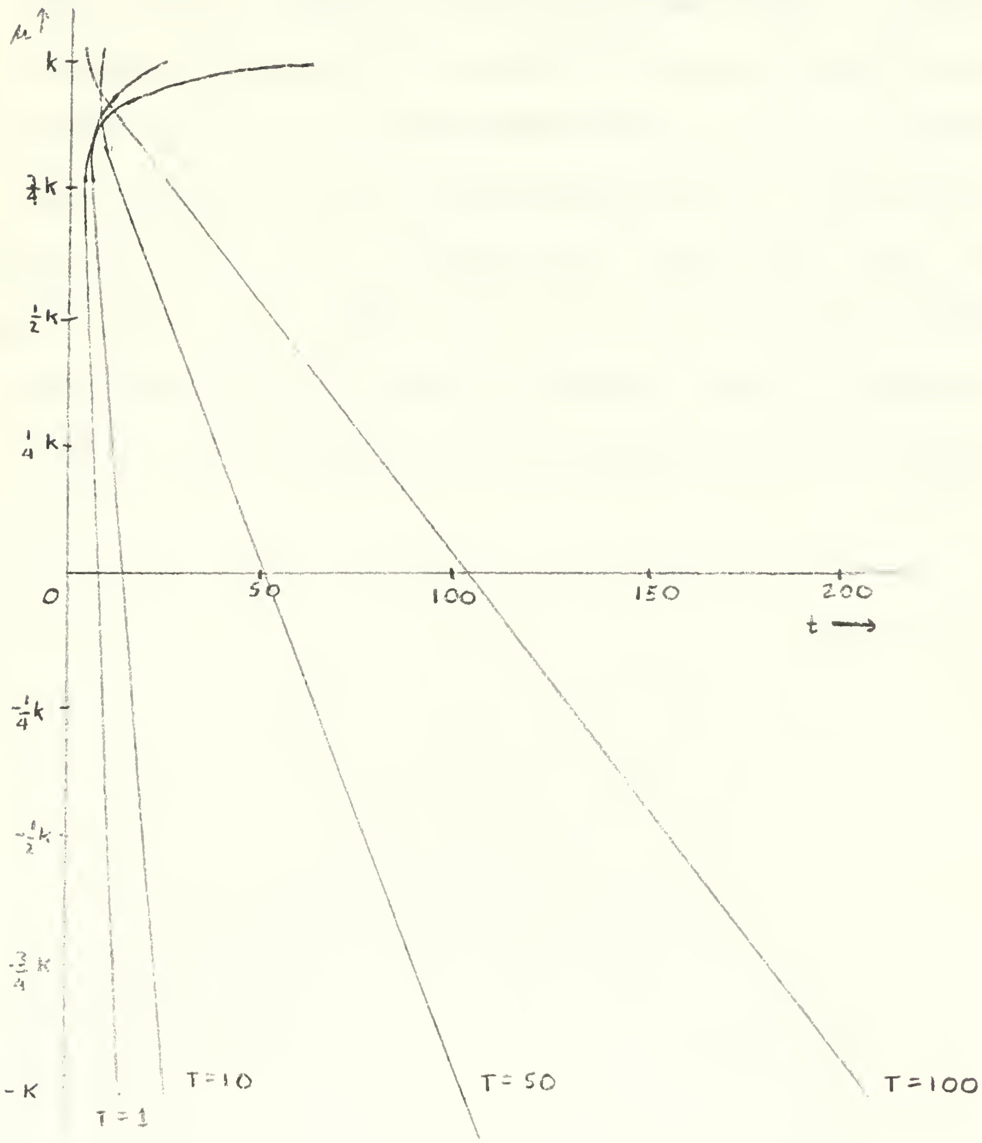

Figure 21. Output pulse width as a function of carrier modulation rate.

It is evident from the curves that by proper pulse shaping of matched inputs side lobe supression and hence ambiguity minimization, may be achieved, but only at the expense of compression, thereby losing resolution. It is also evident that mismatching of the input causes large deviations from predicted output, with greatly reduced pulse compression. Pulse stretching of even long pulses can be achieved if the mismatch is severe enough. E.g. figure (11d) shows pulse expansion of two to one when the input is chirping in the opposite sense. The output pulse width appears linearly related to the carrier modulation rate except in the neighborhood of matched conditions.

4. Recommendations For Future Work.

In this report, the shape of the output envelope for various matched and mismatched envelopes has been derived and plotted. While analytical expressions for the output phase were indicated, there was no attempt to analyze or to display this information graphically. This phase structure may also be of interest, and hence warrant additional work. For example, the problem of indentifying and processing doppler shift information could be explored by a careful analysis of the phase structure of the filter output signal.

In addition, it is of interest to inquire about the output of a dispersive delay filter when the input is a random process. This output would necessarily have to be described statistically. In general, the power spectrum of the output from a linear filter, in response to a random process input, is equal to the power spectrum of the input process times the square of the magnitude of the filter transfer function [8]. I.e.

$$G_{out}(\omega) = G_{in}(\omega)|H(\omega)|^2$$

For the ideal dispersive delay filter

$$H(\omega) = e^{j\frac{(\omega-\omega_0)^2}{2K}}$$

and

$$|H(\omega)|^2 \equiv 1$$

Therefore the power spectrum of the output is identically equal to the power spectrum of the input. Another statistic that may be meaningful in some applications is the form to the output amplitude density function. The problem is then to evaluate the amplitude density function of

$$g(t) = \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} f(\tau)\, e^{jkt\tau}\, d\tau$$

where $g(t)$, $f(\tau)$, and k are as previously defined, except that now $f(\tau)$ is a random process of known amplitude density function. For an input random process which is gaussian, Davenport and Root [8] have shown g(t) will also have a gaussian distribution. However, Papoulis [9] has indicated that there are other classes of random processes, which could yield a gaussian output process. An examination of the above problem could yield meaningful results, for example when considering applications of these dispersive delay lines to communications, data processing, and measurement systems.

In a recent article, Ward [15] has examined a pulse compression filter with arbitrary time duration impulse response. In particular, he has derived an output expression for such a filter driven by a matched rectangular pulse input. Additional work in analysis of this band limited filter for arbitrary matched and mismatched inputs, would be of interest.

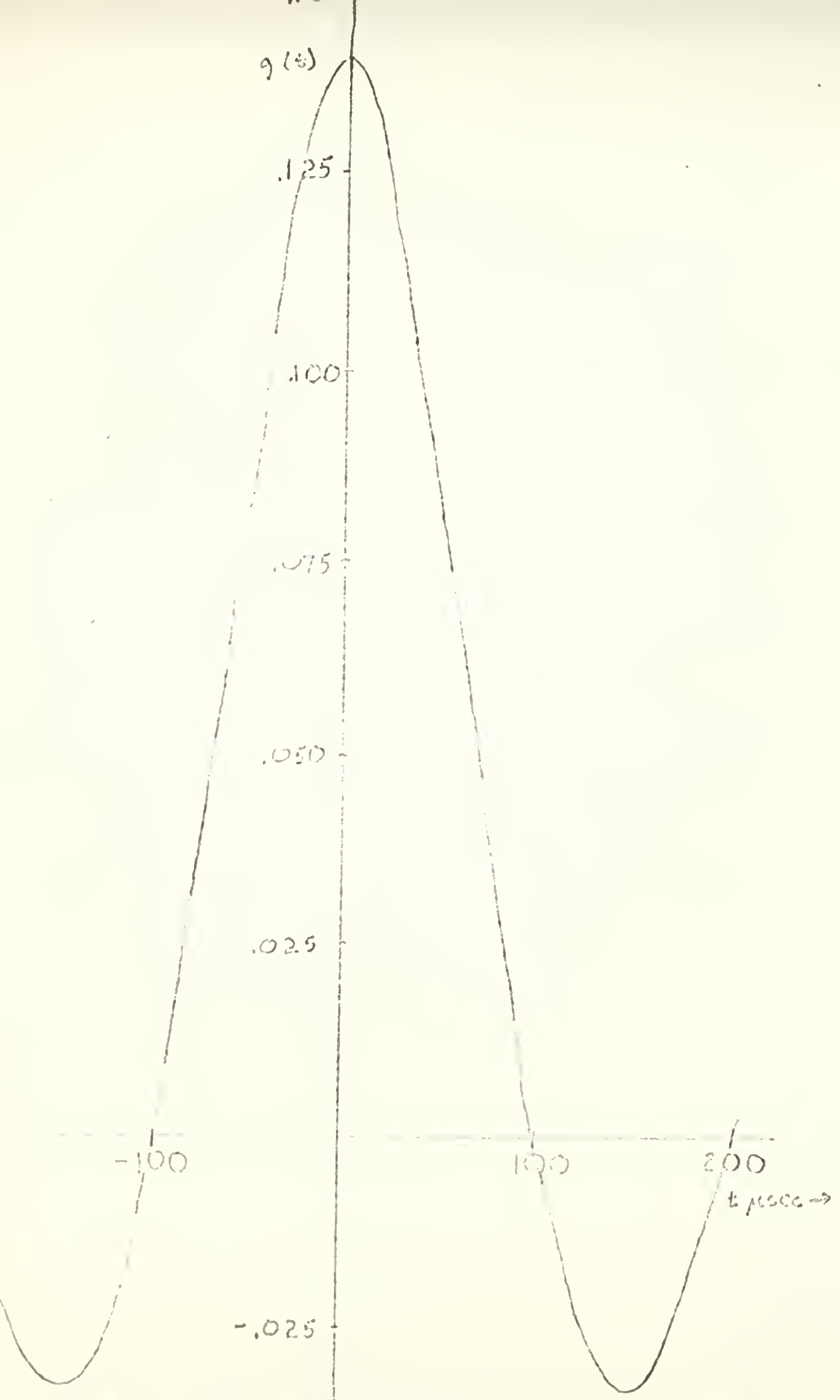

Figure 4(a) Matched Rectangular Input, T = 1 μsec

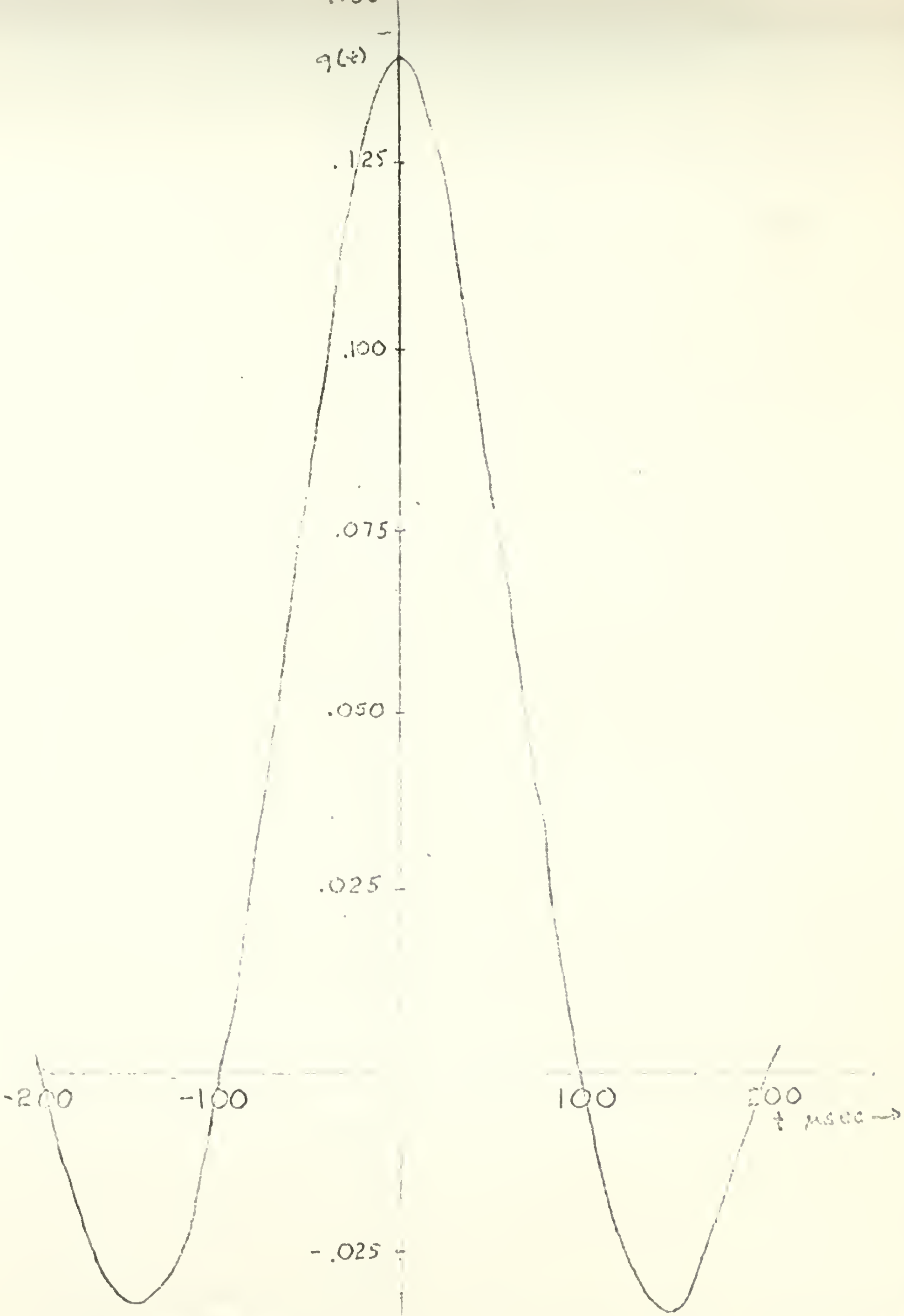

Fig 5a Constant Carrier Frequency Rectangular Input, T = 1 μsec

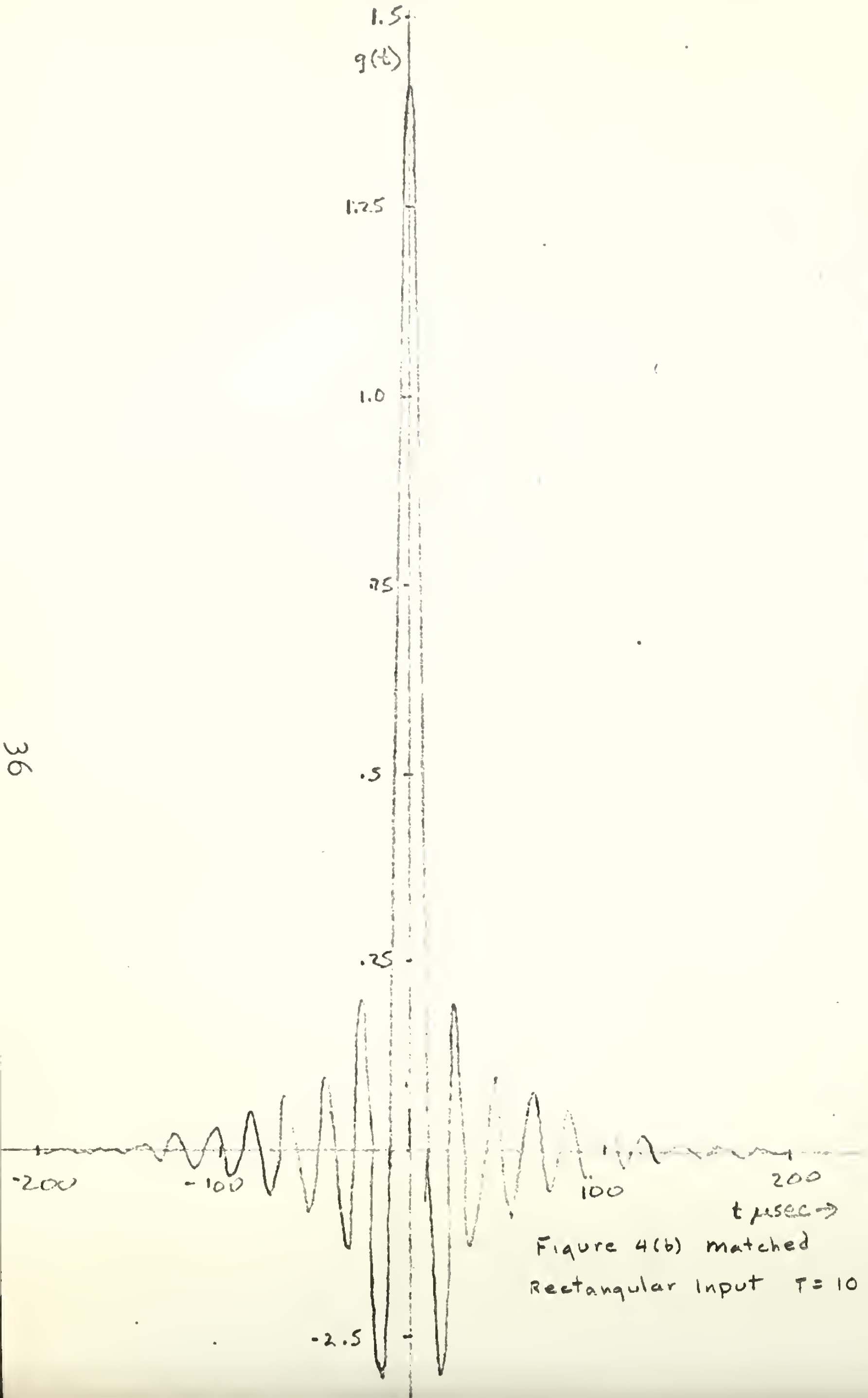

Figure 4(b) Matched
Rectangular Input T = 10 μsec

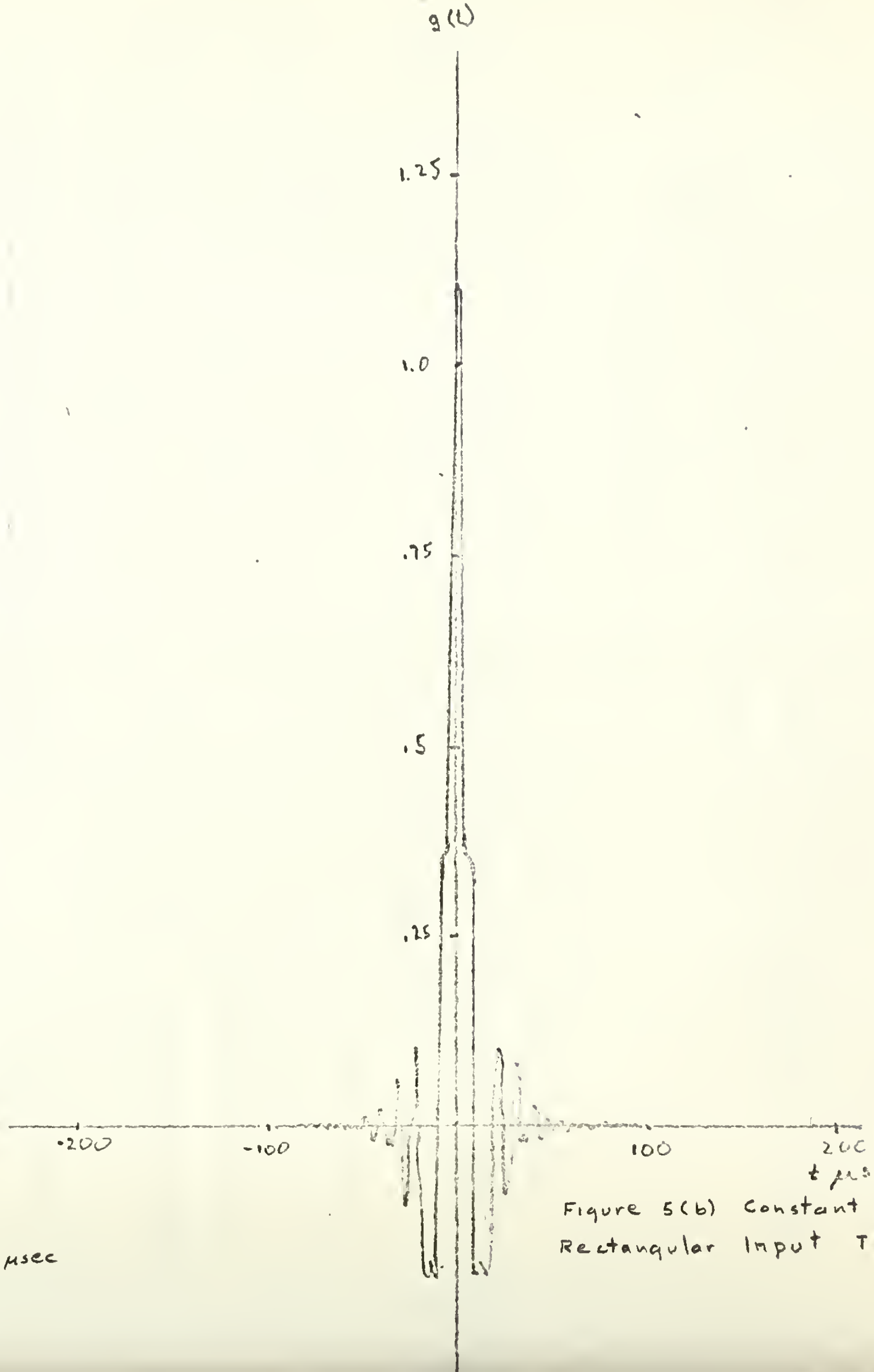

Figure 5(b) Constant Carrier
Rectangular Input T = 10 μsec

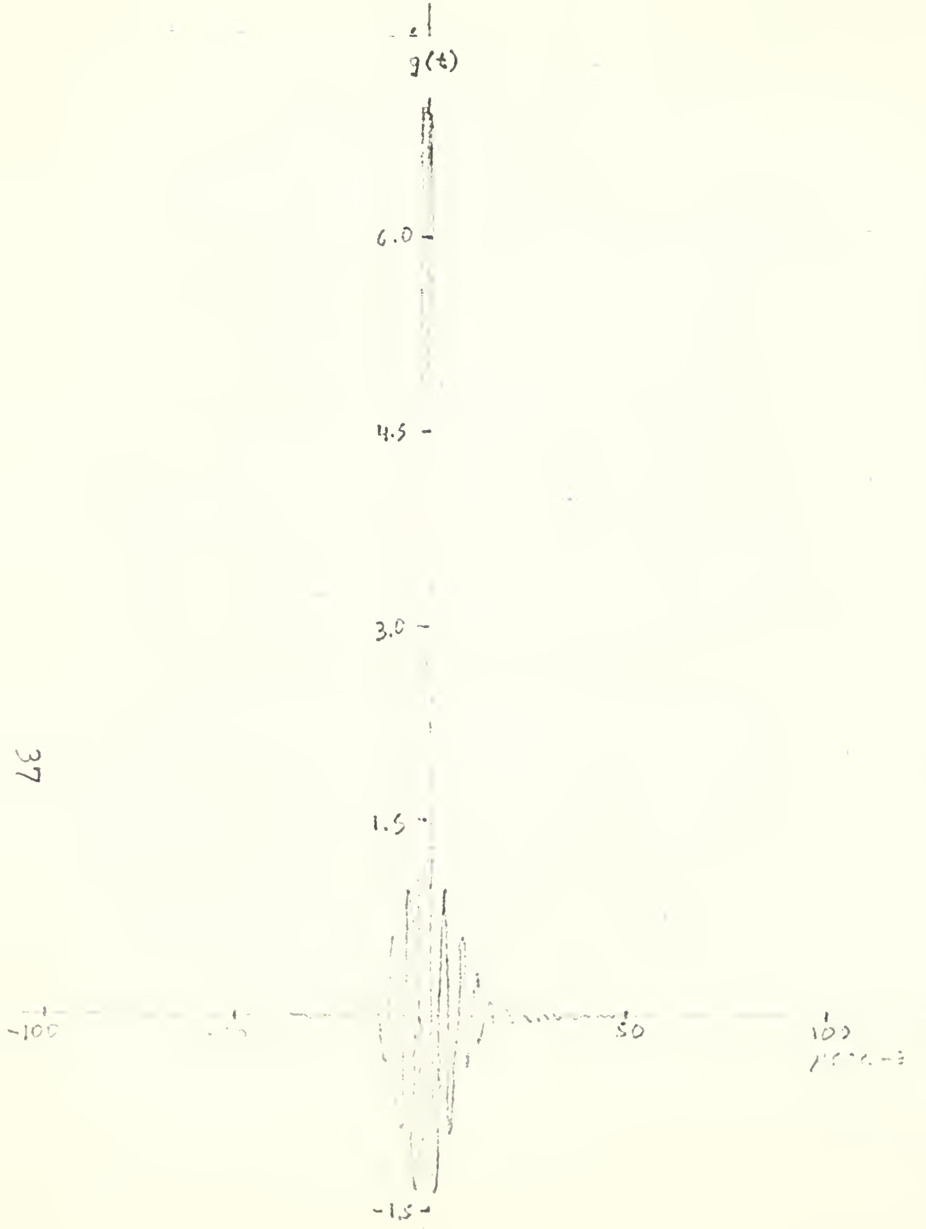

Figure 4(c) Matched Rectangular Input T= 50 μsec

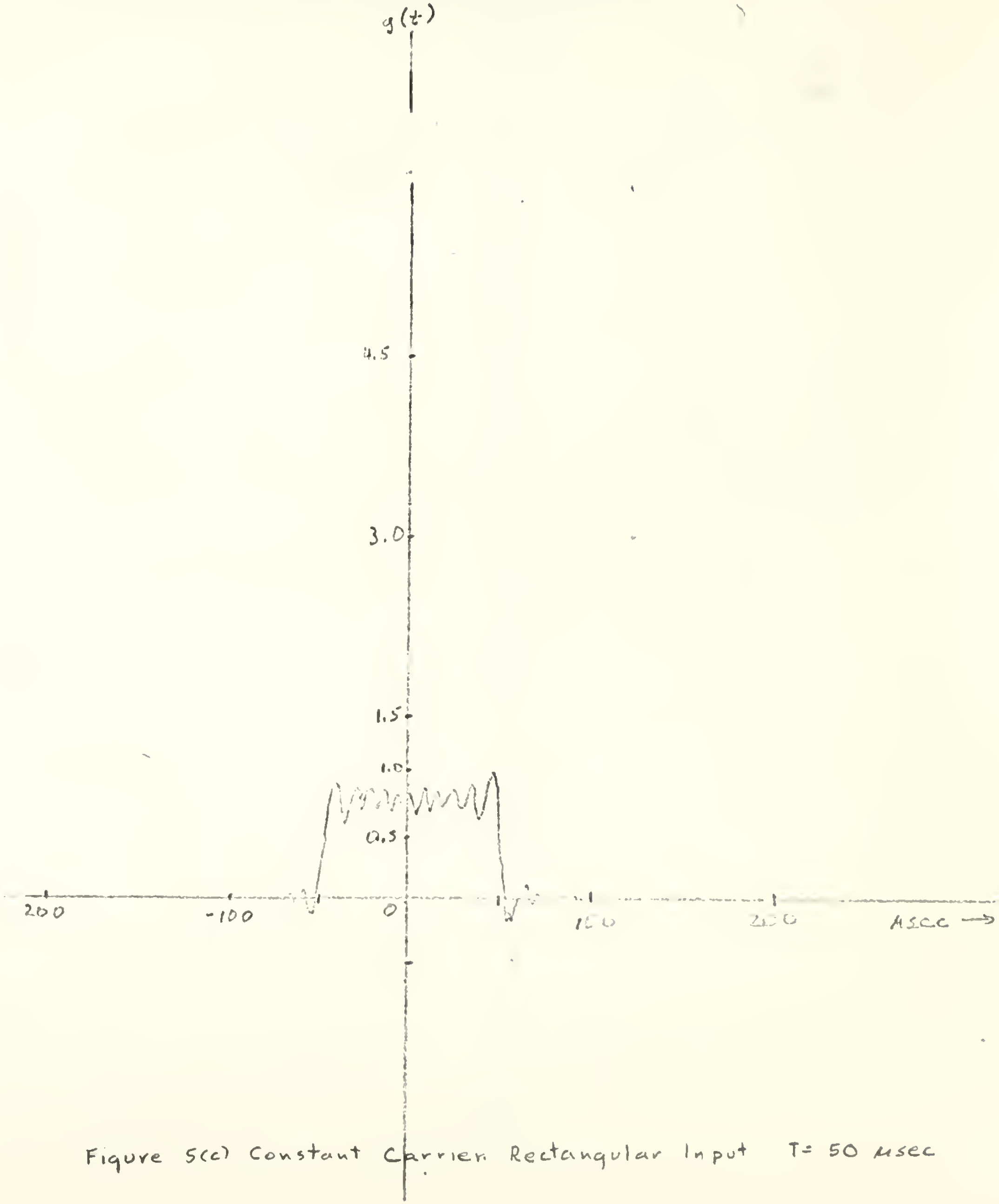

Figure 5(c) Constant Carrier Rectangular Input T= 50 μsec

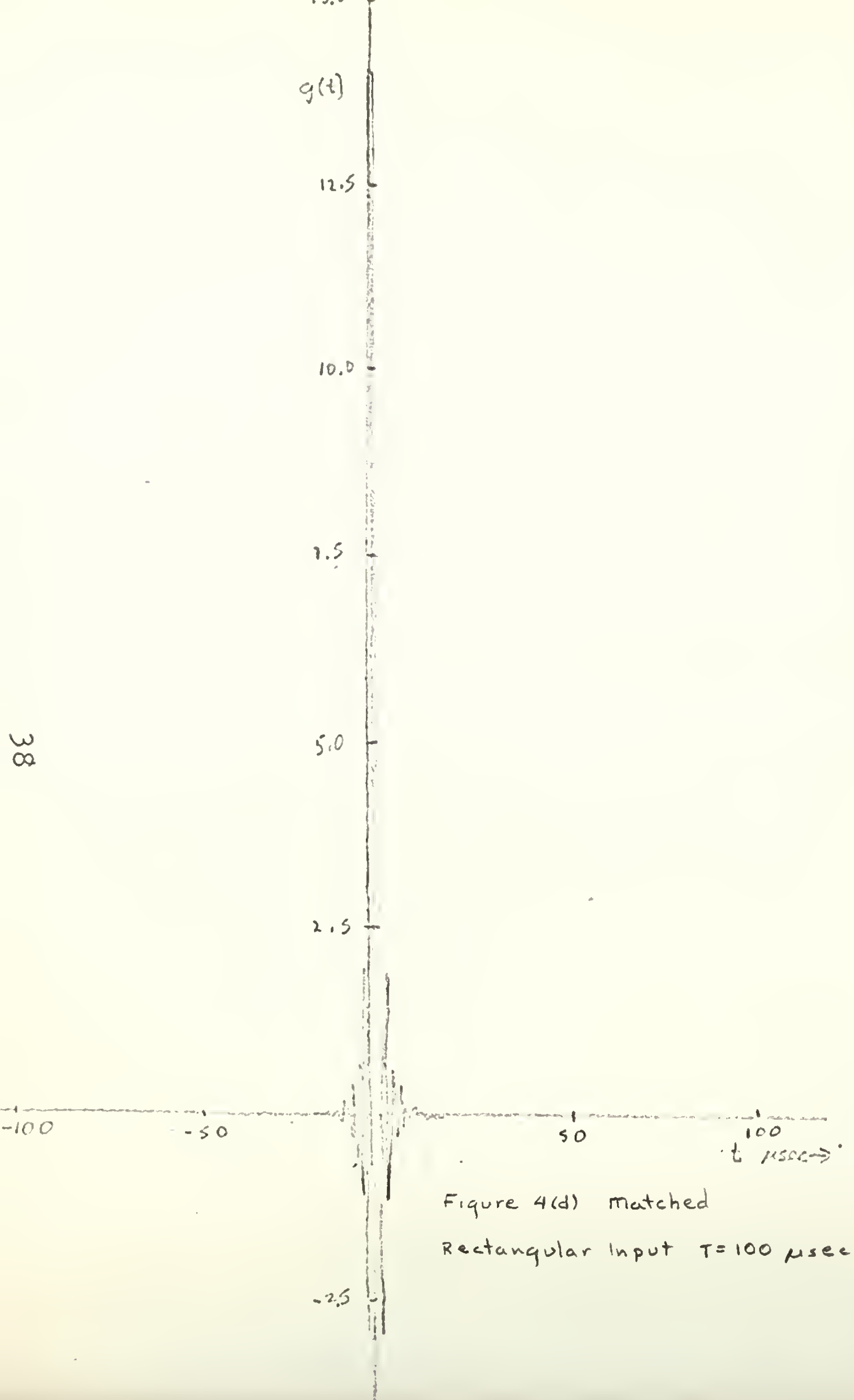

Figure 4(d) matched

Rectangular Input T = 100 μsec

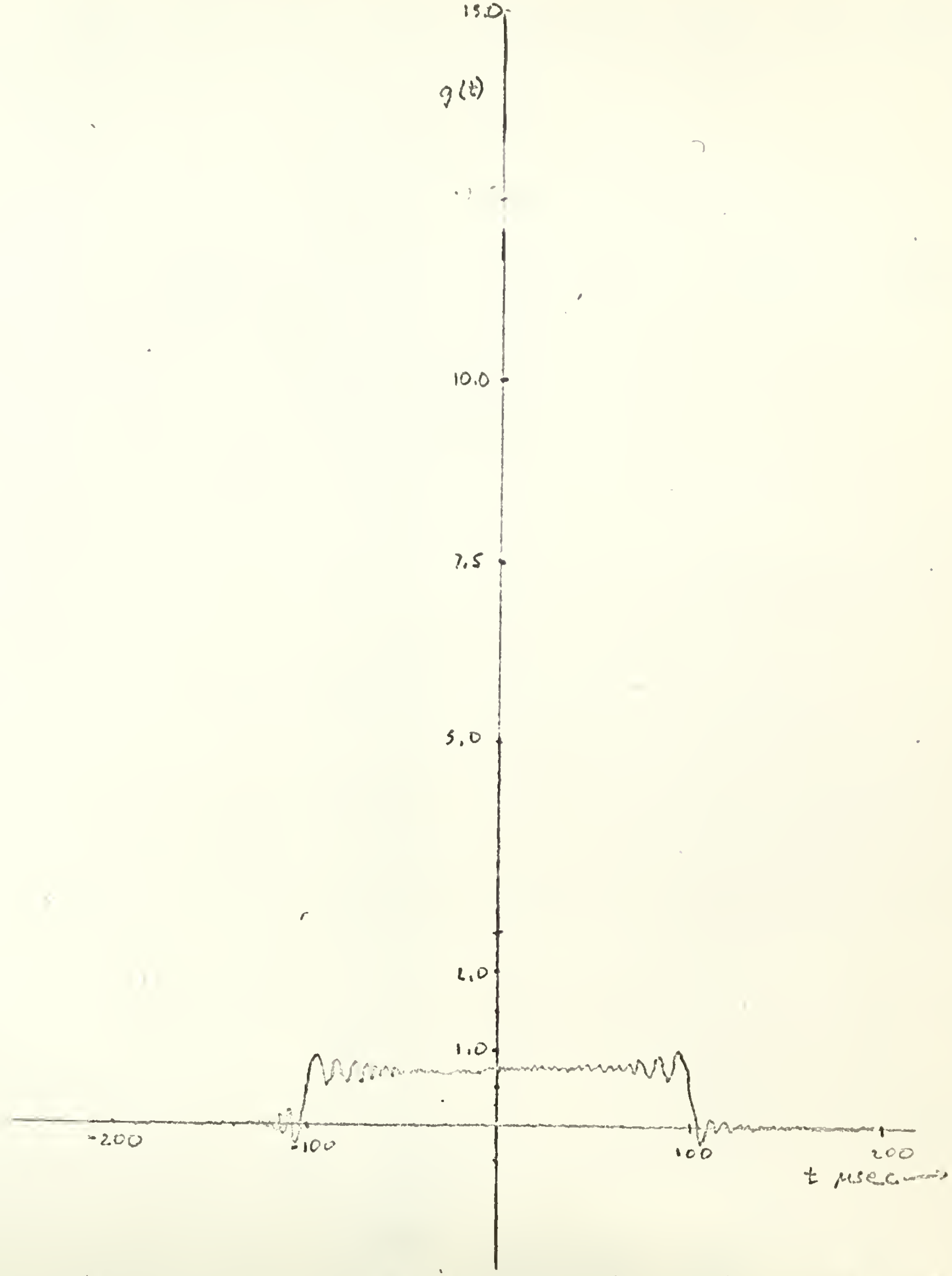

Figure 5(d) Constant Carrier Rectangular Input T = 100 μsec

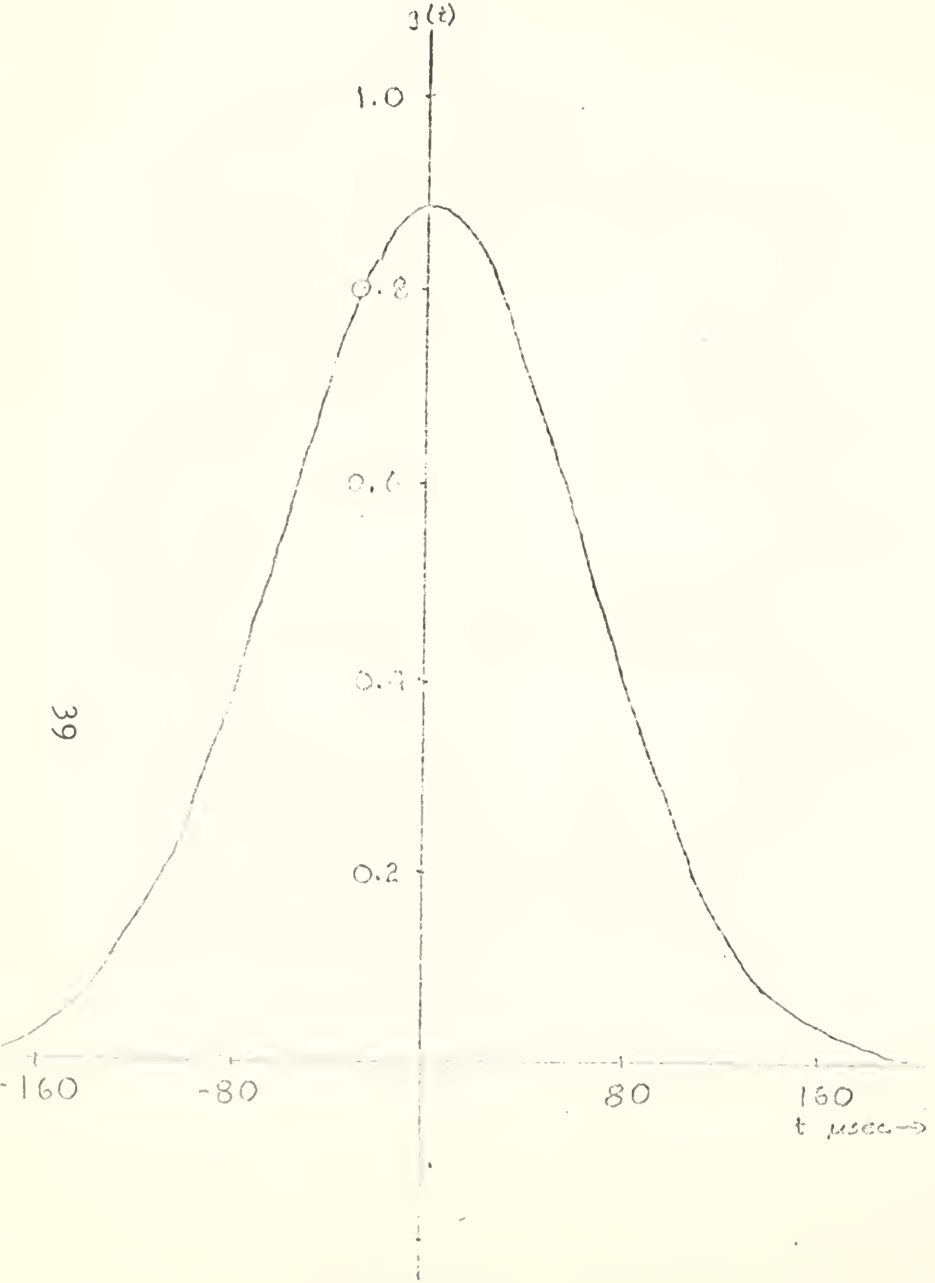

Figure 6(a) Matched Gaussian Input $\bar{e}^{2} = 1\ \mu sec$

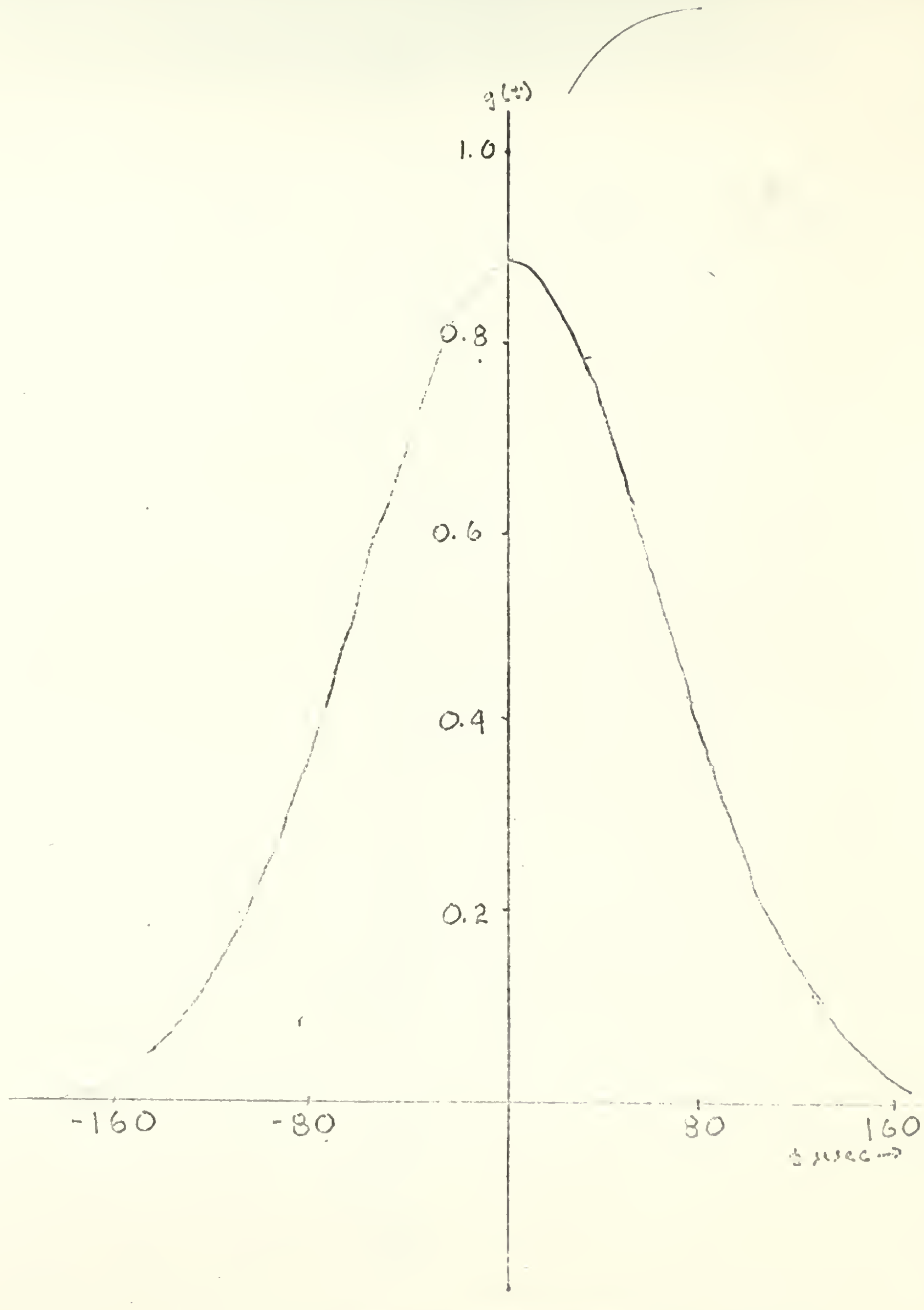

Figure 7(a) Constant Carrier Gaussian Input $\bar{e}^{2} = 1\ \mu sec$

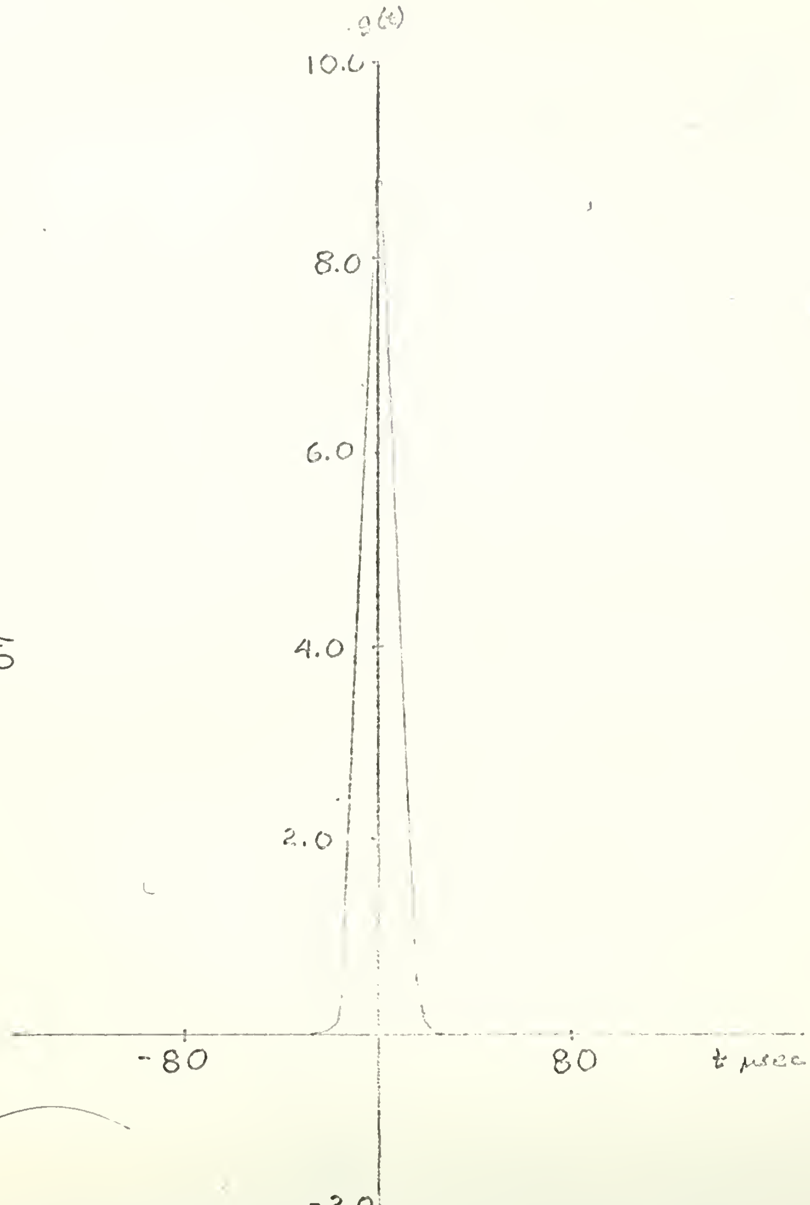

Figure 6(b) Matched Gaussian Input $e^{-2} = 10$ μsec

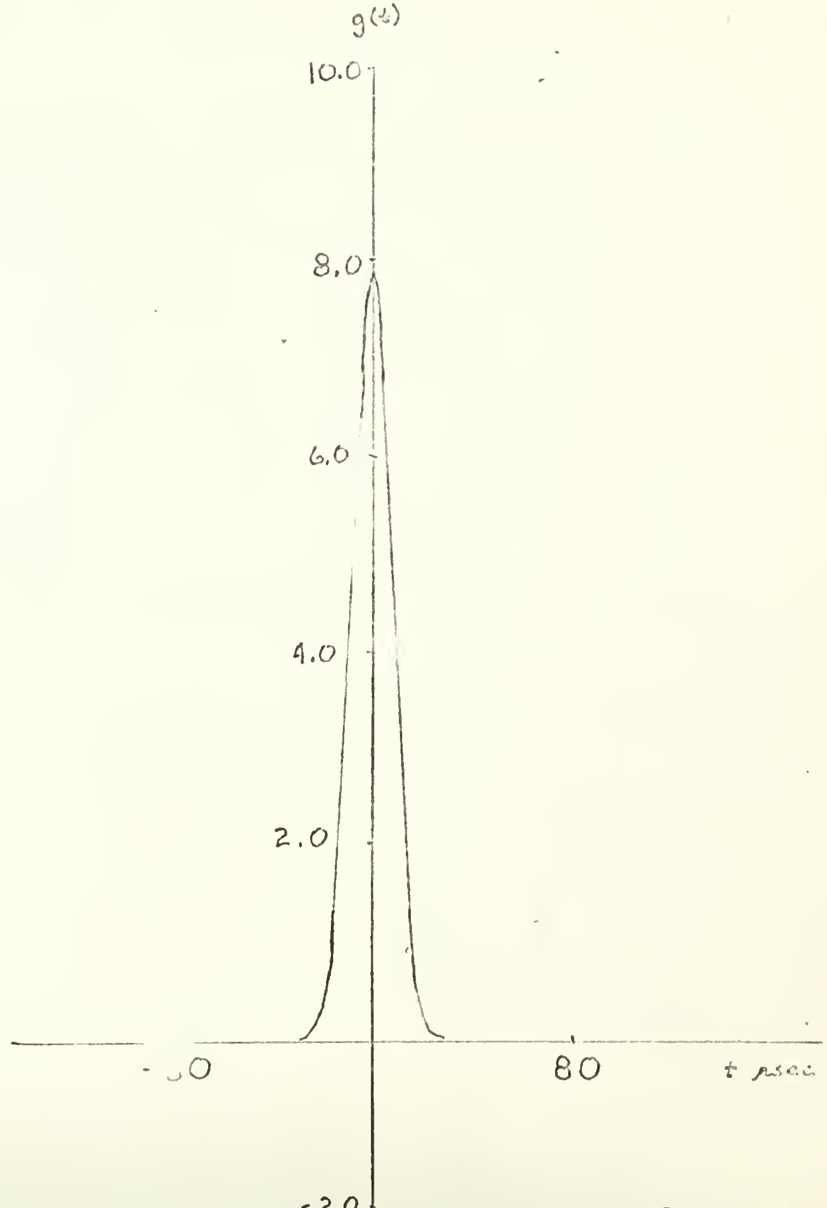

Figure 7(b) Constant Carrier Gaussian Input $e^{-2} = 10$ μsec

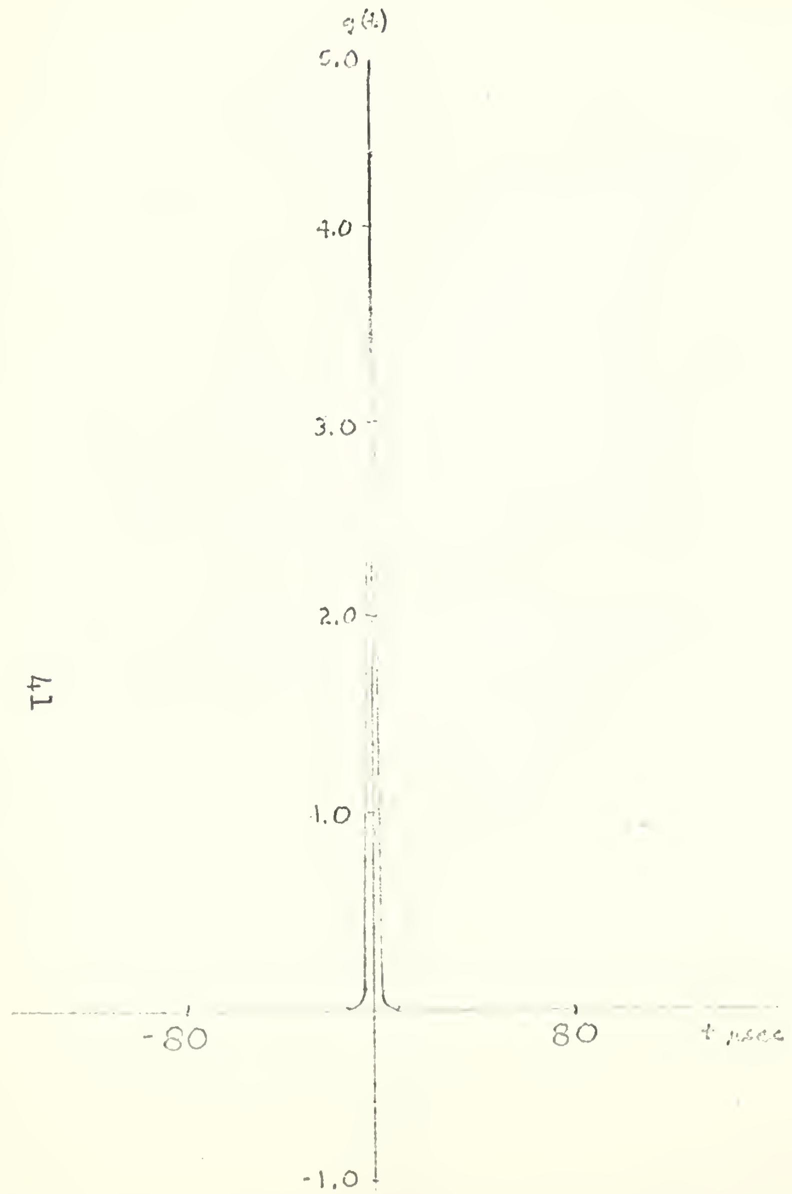

Figure 6 (c) Matched Gaussian Input $e^{-2} = 50$ μsec

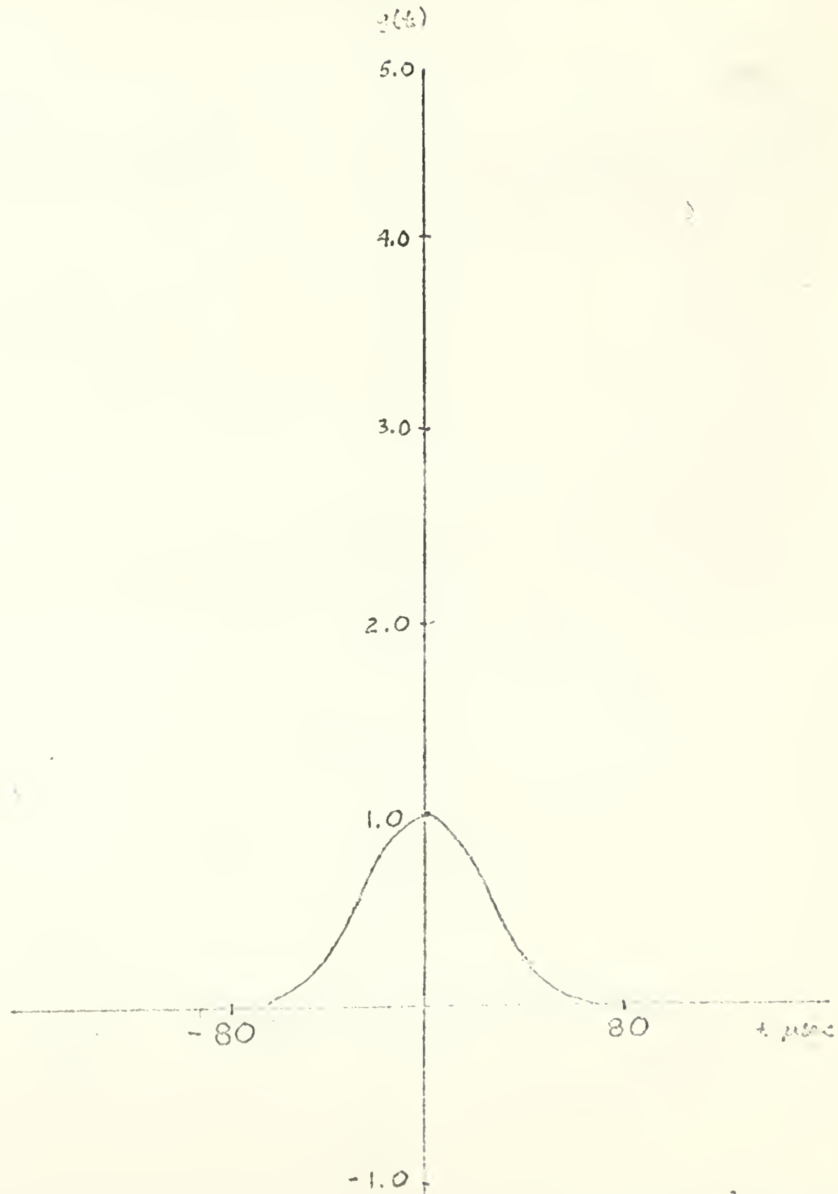

Figure 7(c) Constant Carrier Gaussian Input $e^{-2} = 50$ μsec

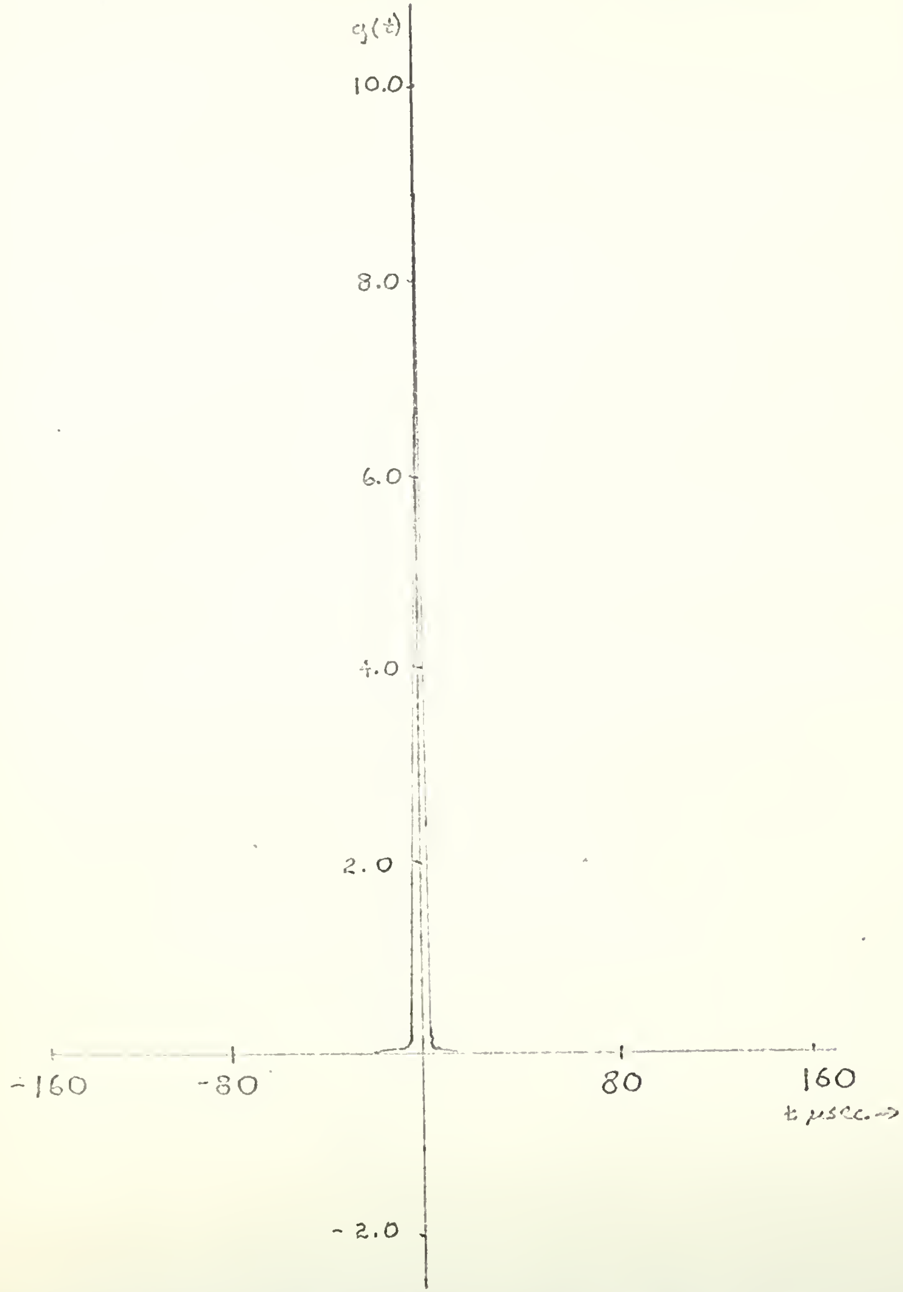

Figure 6(d) Matched Gaussian Input $e^{-2} = 100\ \mu sec$

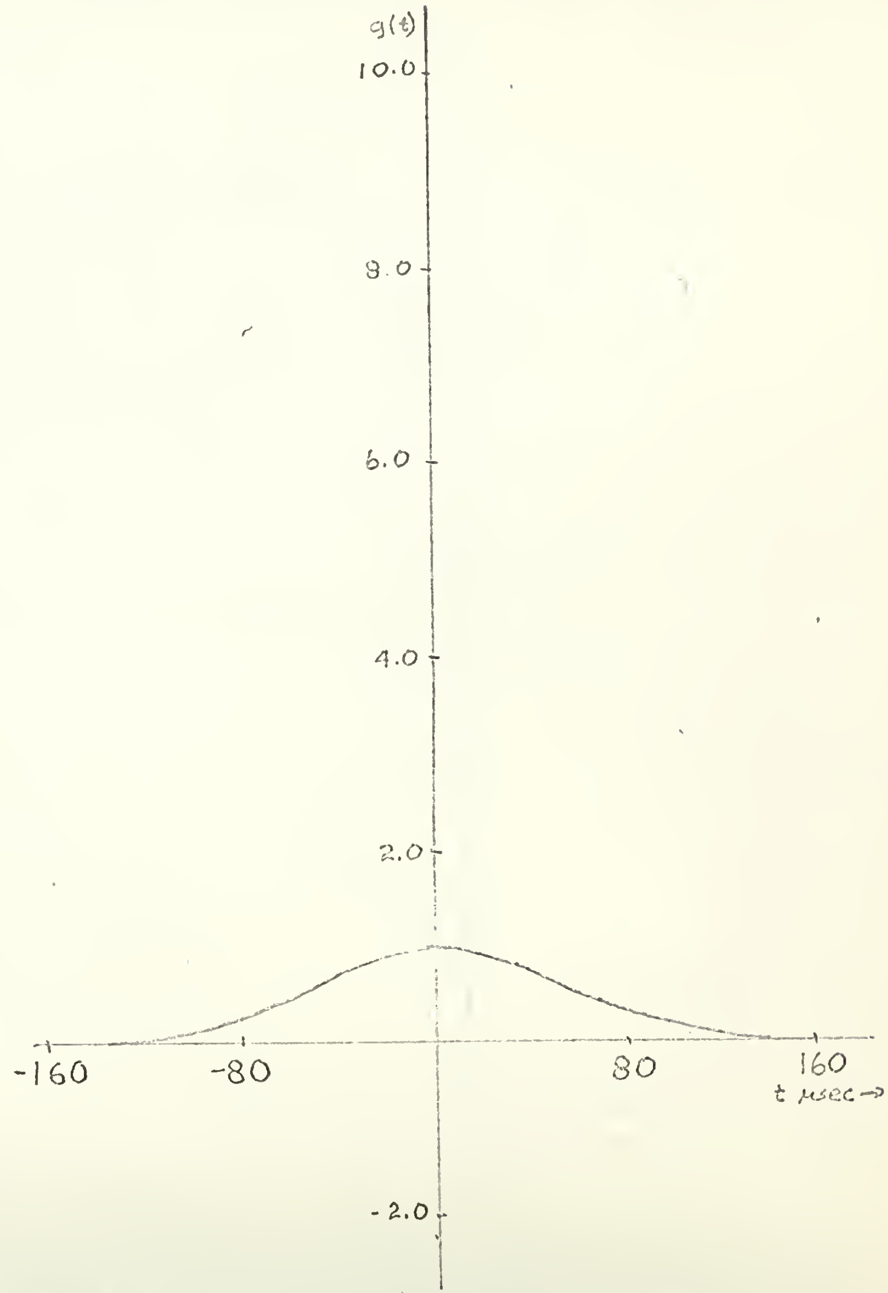

Figure 7(d) Constant Carrier Gaussian Input $e^{-2} = 100\ \mu sec$

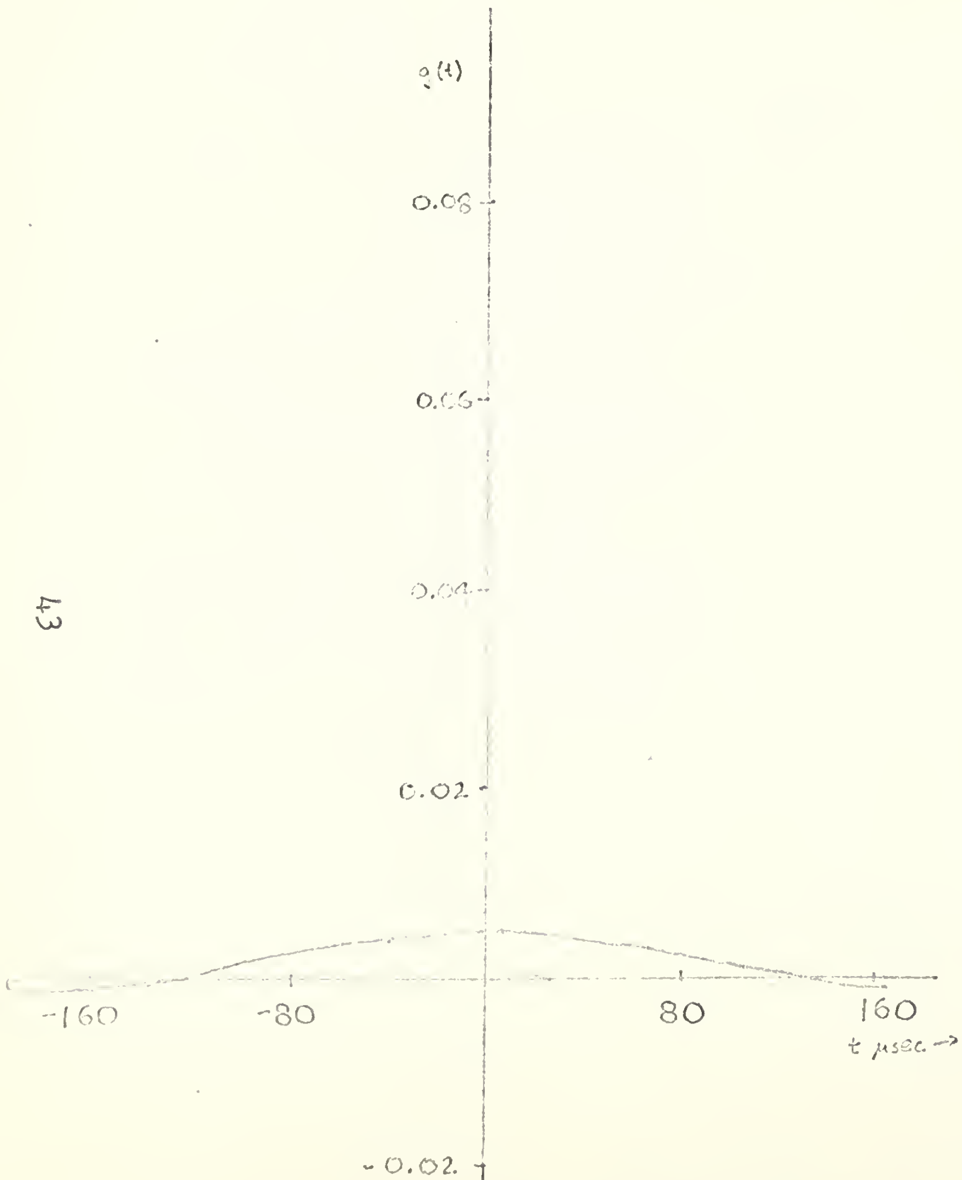

Figure 8(a) Matched Half Cosine Input $F_1 = 250$ kc

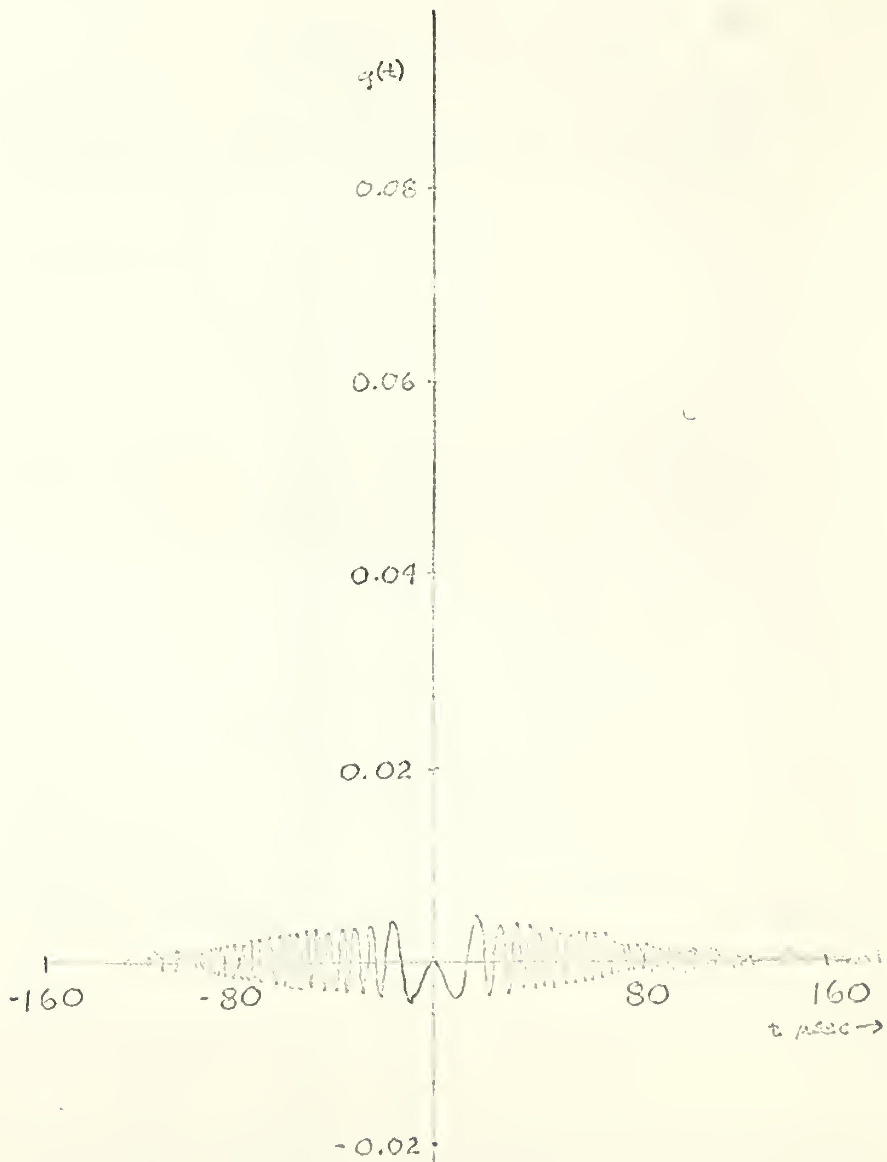

Figure 9(a) Constant Carrier Half Cosine Input $F_1 = 250$ kc

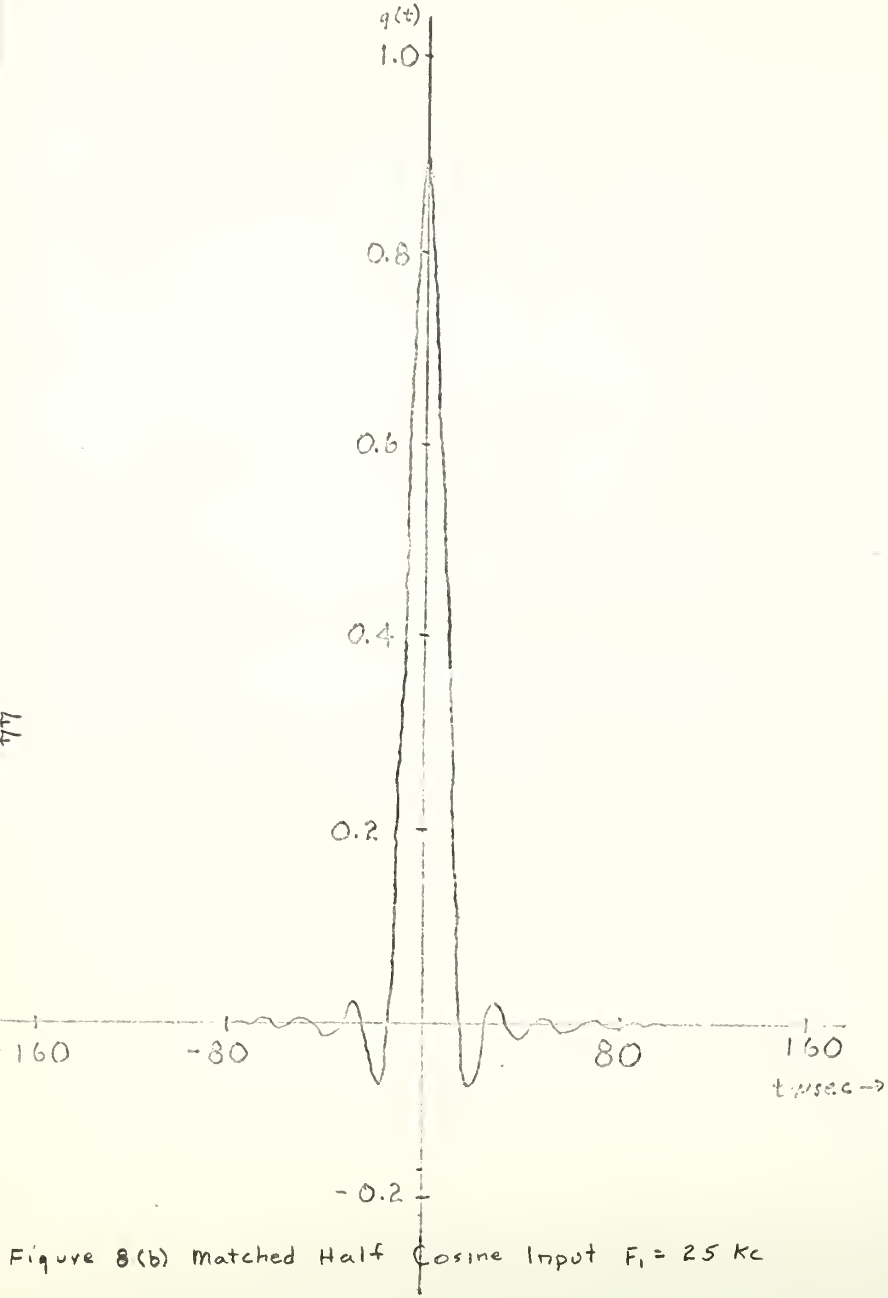

Figure 8(b) Matched Half Cosine Input $F_1 = 25$ Kc

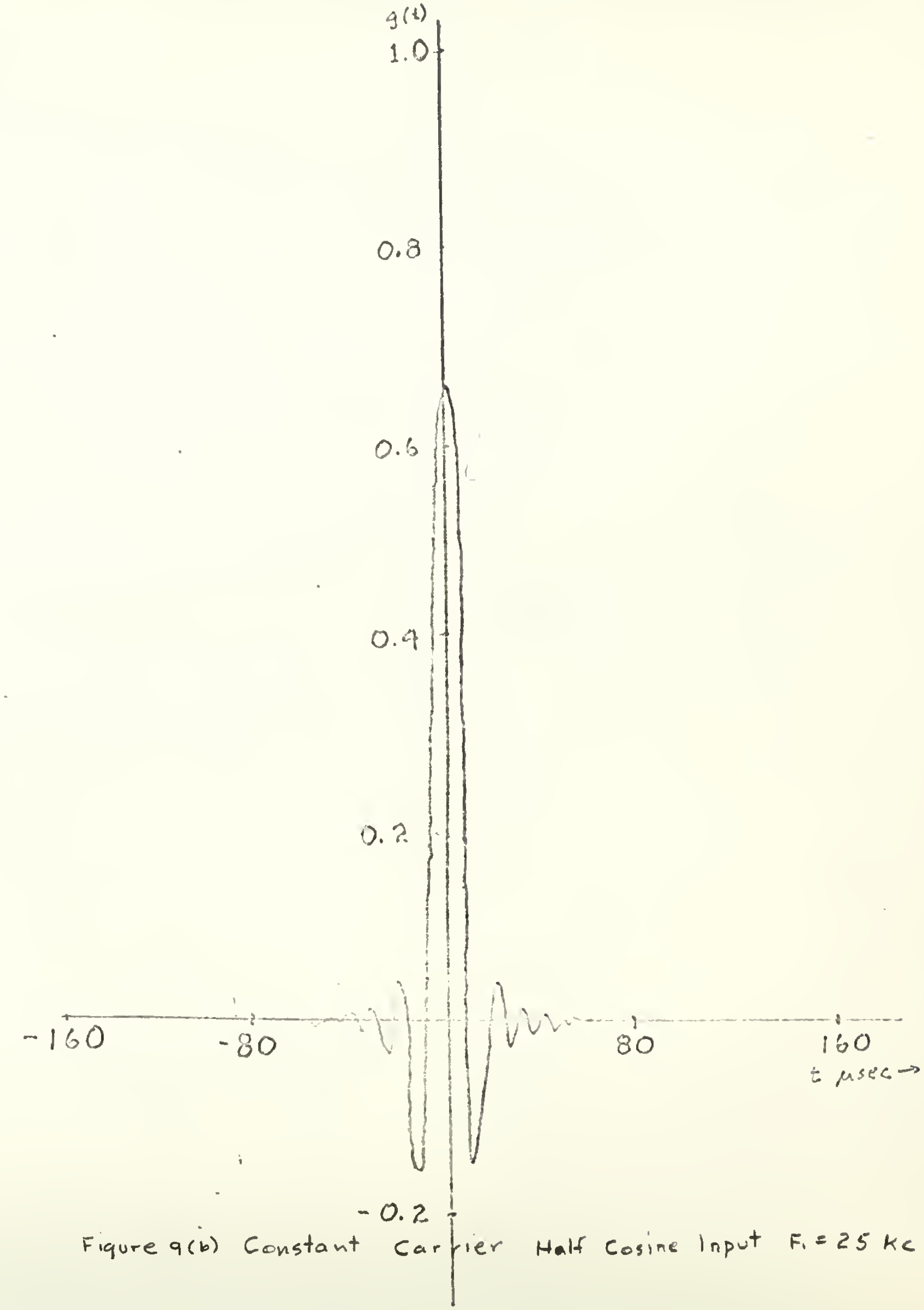

Figure 9(b) Constant Carrier Half Cosine Input $F_1 = 25$ Kc

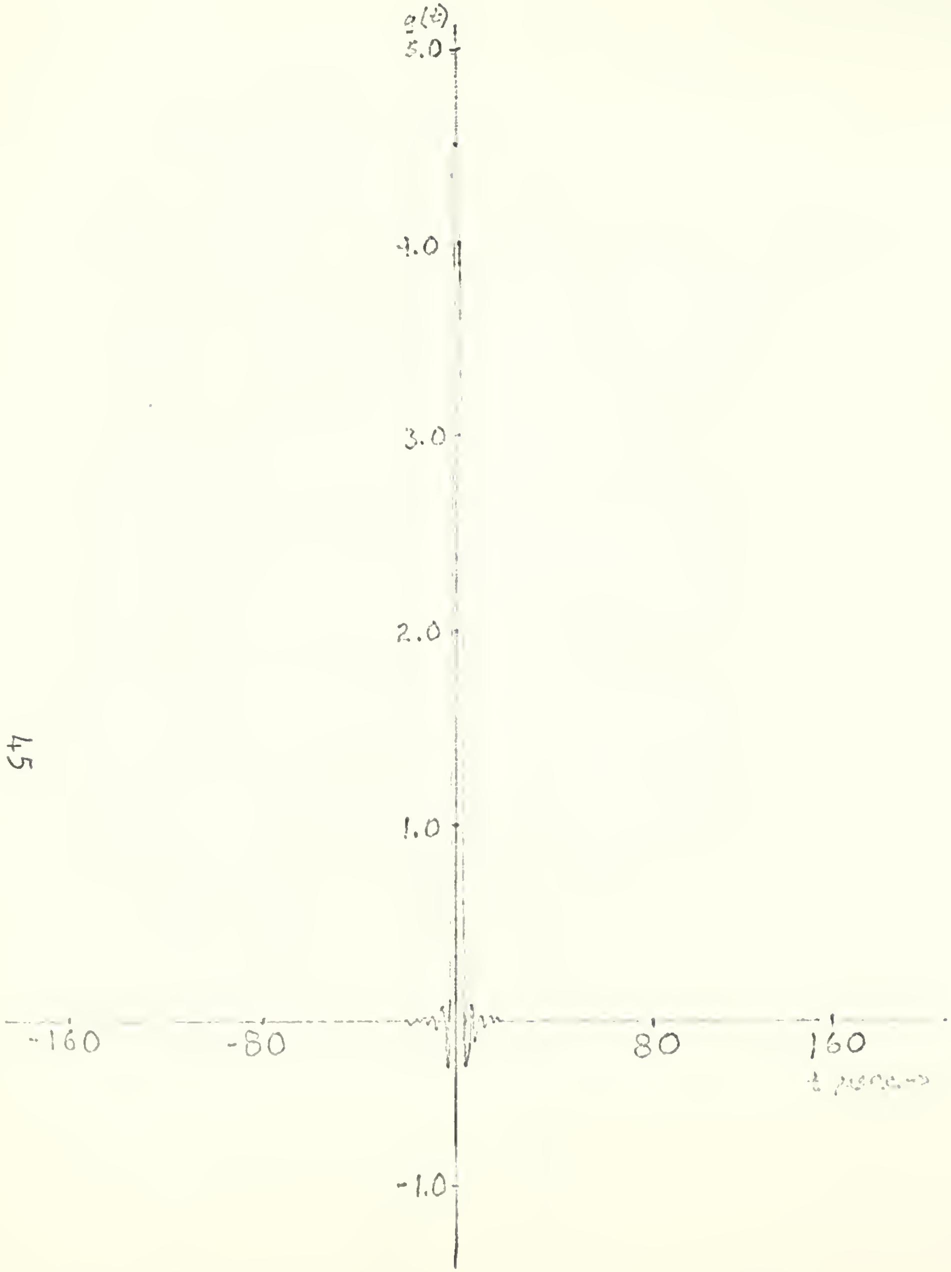

Figure 8(c) Matched Half Cosine Input $F_1 = 5.0$ kc

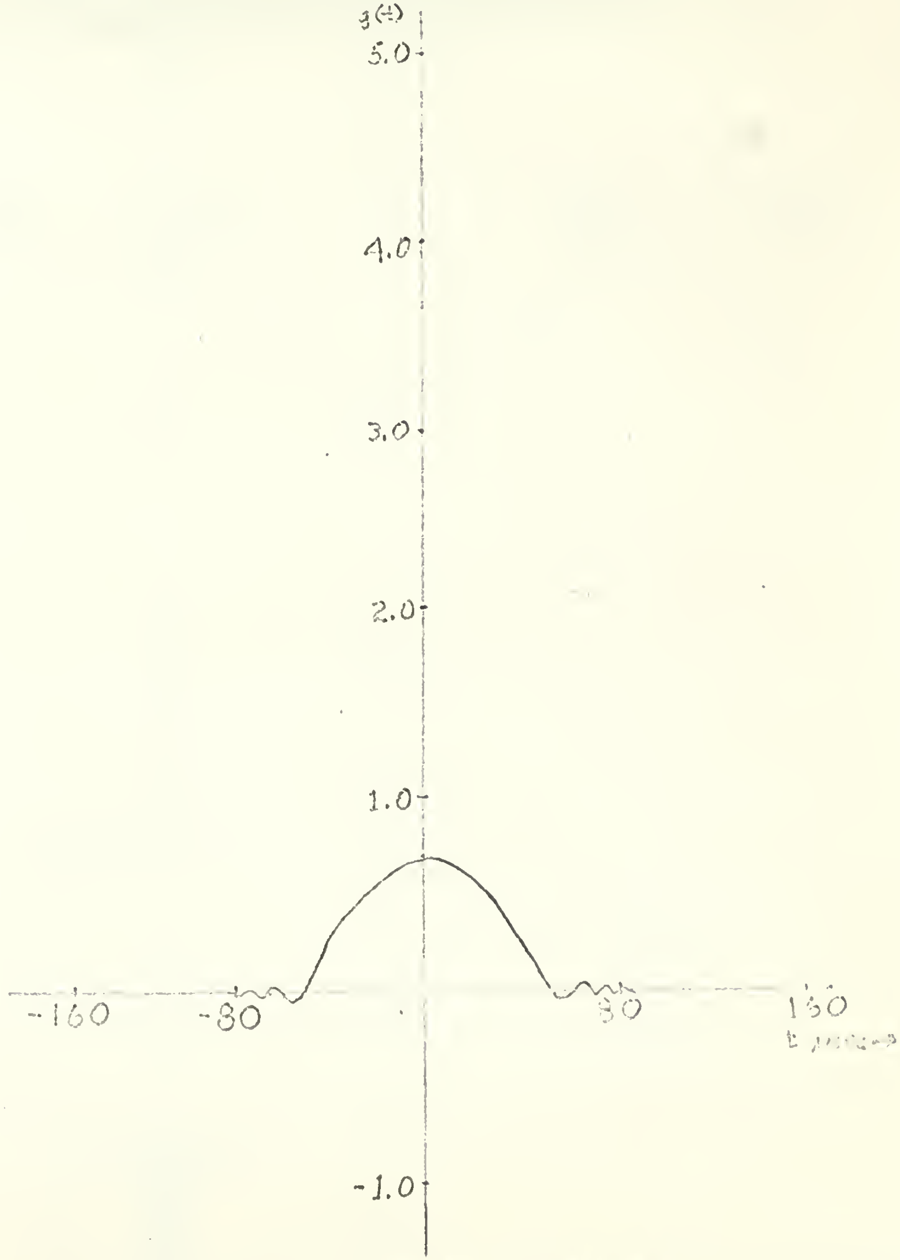

Figure 9(c) Constant Carrier Half Cosine Input $F_1 = 5.0$ Kc

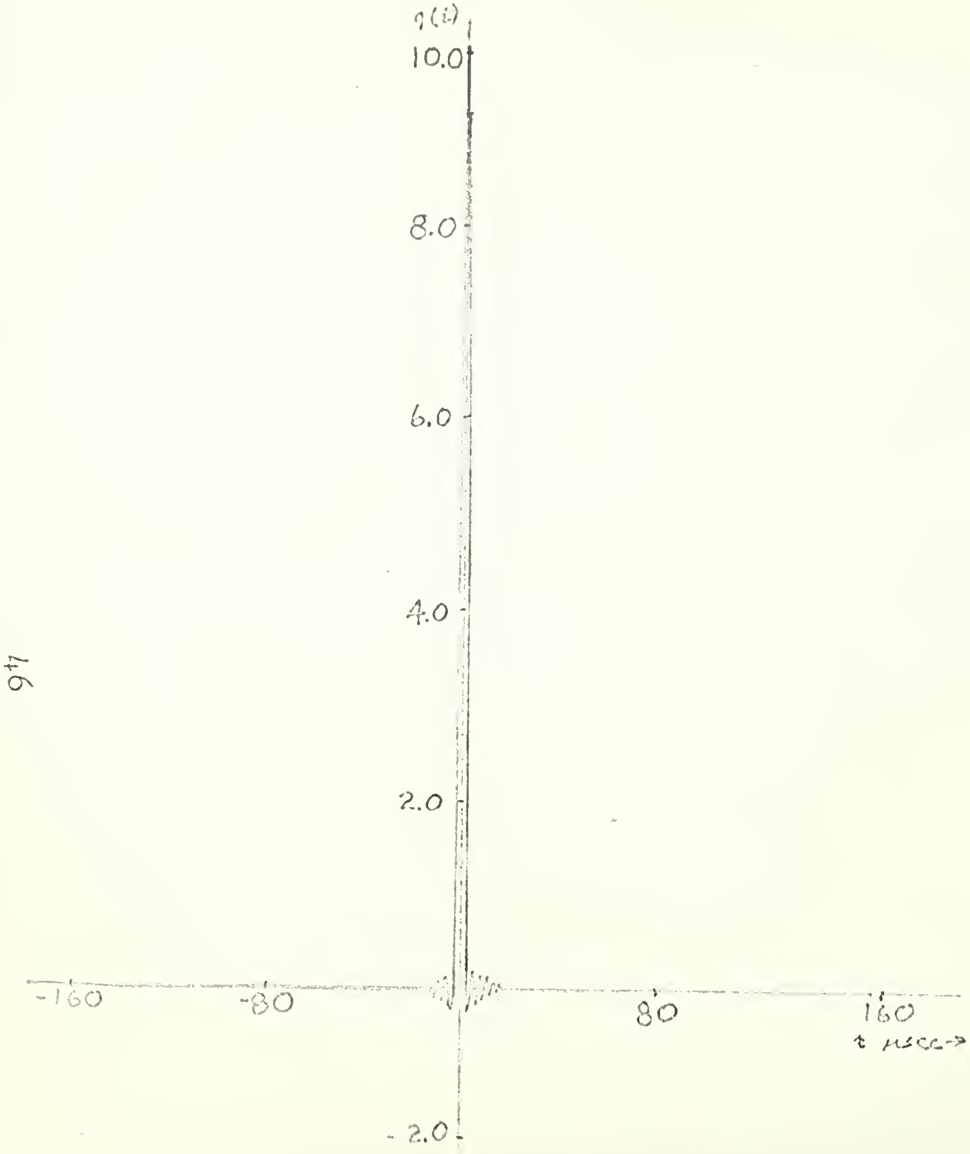

Figure 8(d) Matched Half Cosine Input $F_1 = 2.5$ kc

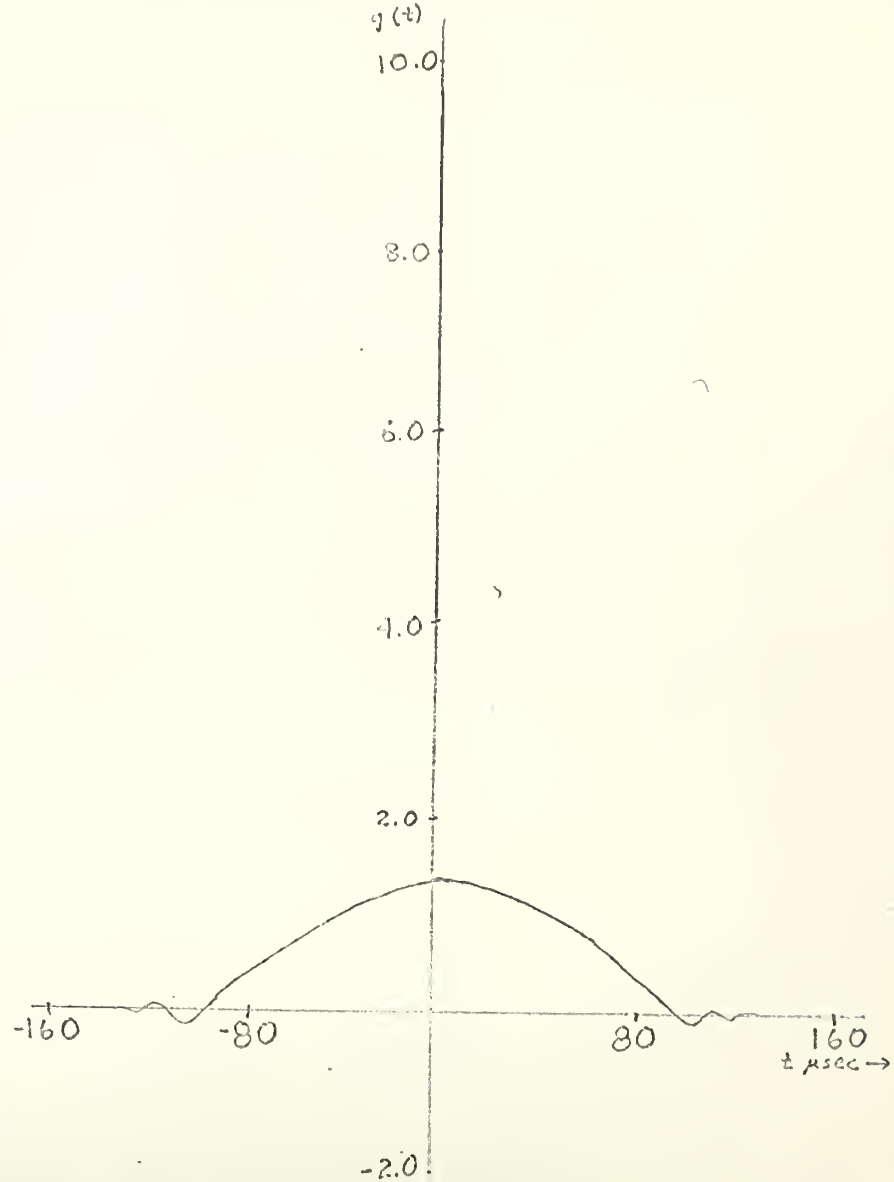

Figure 9(d) Constant Carrier Half Cosine Input $F_1 = 2.5$ kc

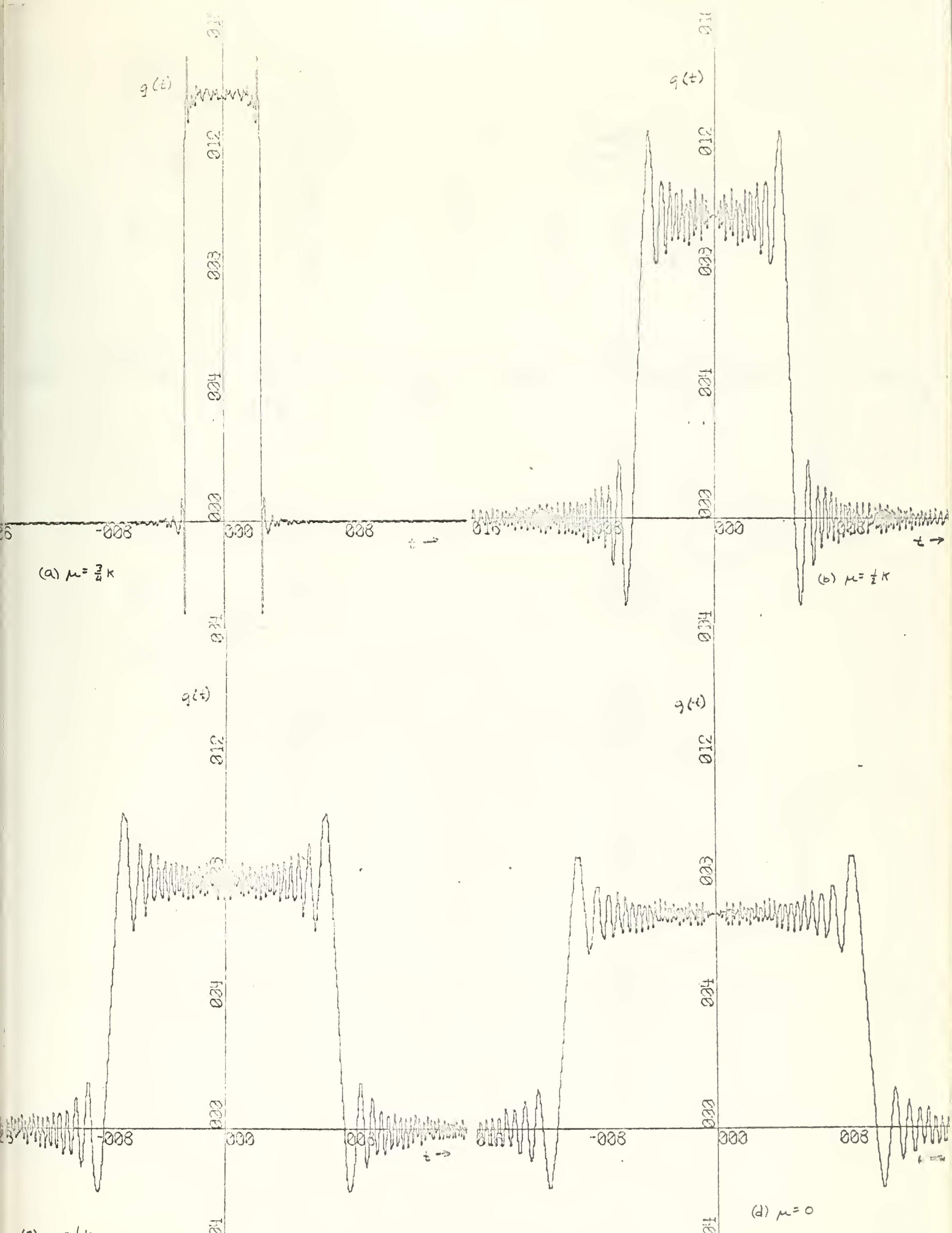

Figure 10 FM Carrier $T = 100$ $\mu > 0$

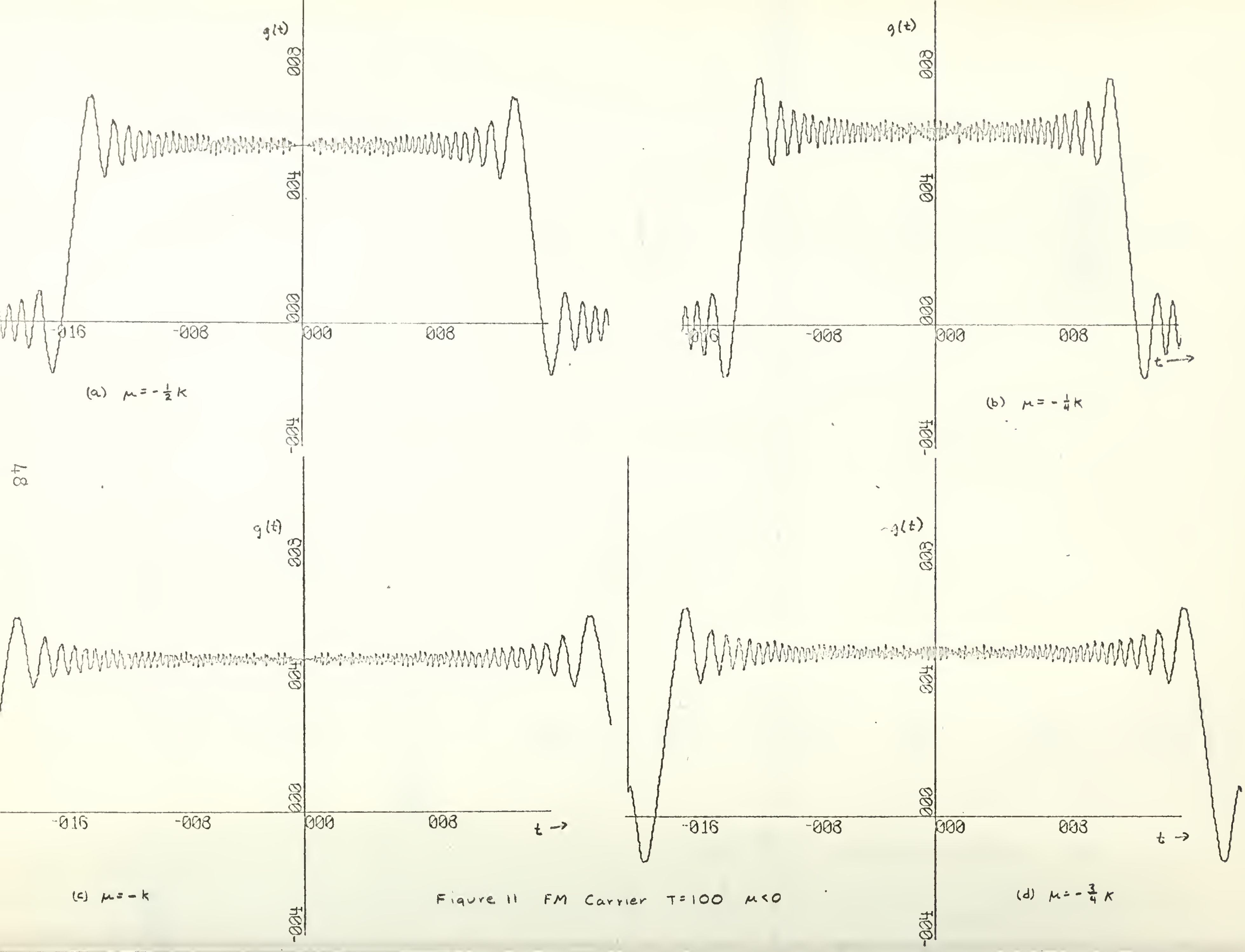

Figure 11 FM Carrier T=100 $\mu < 0$

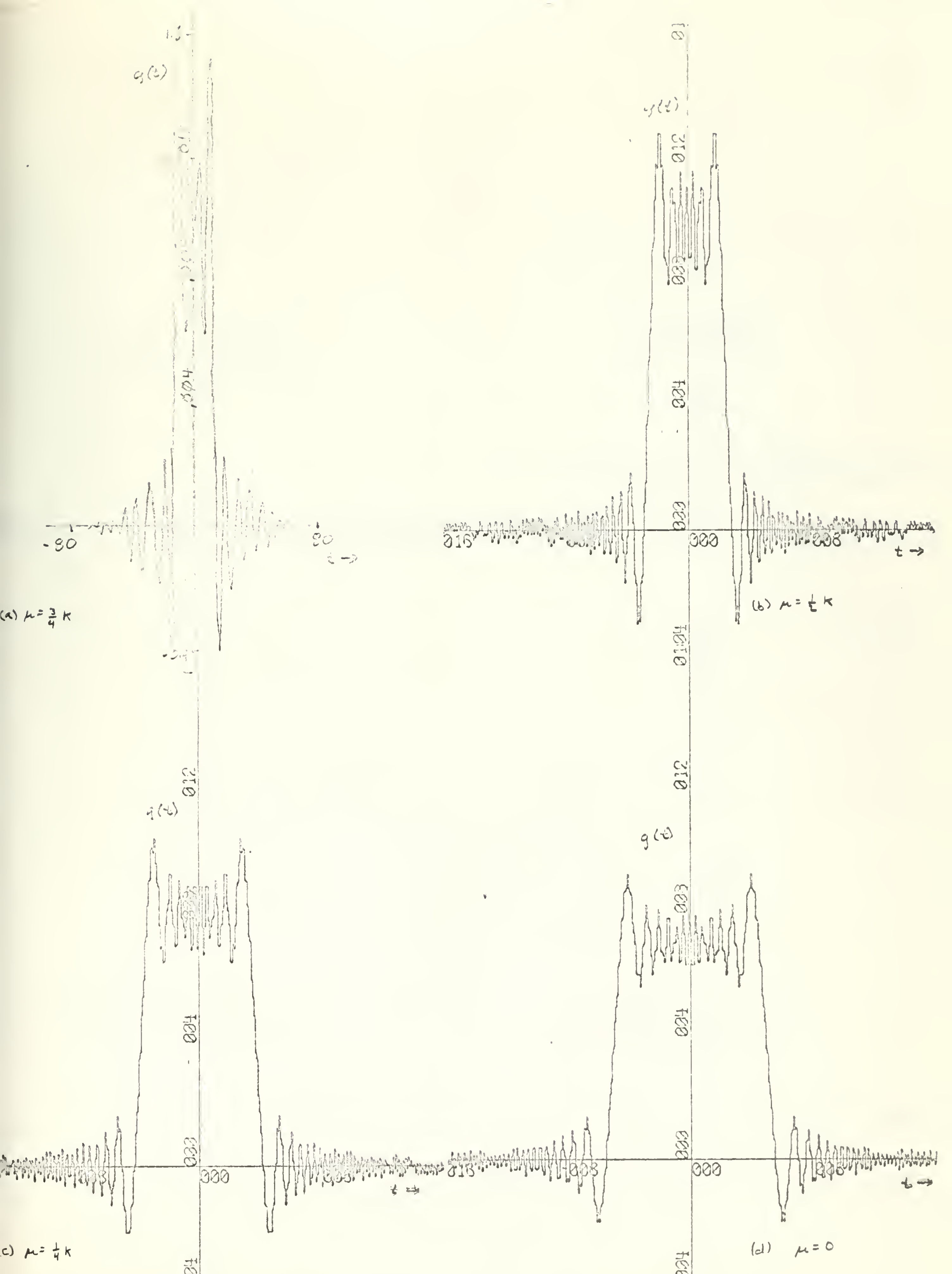

Figure 12 FM Carrier $T = 50$ $\mu > 0$

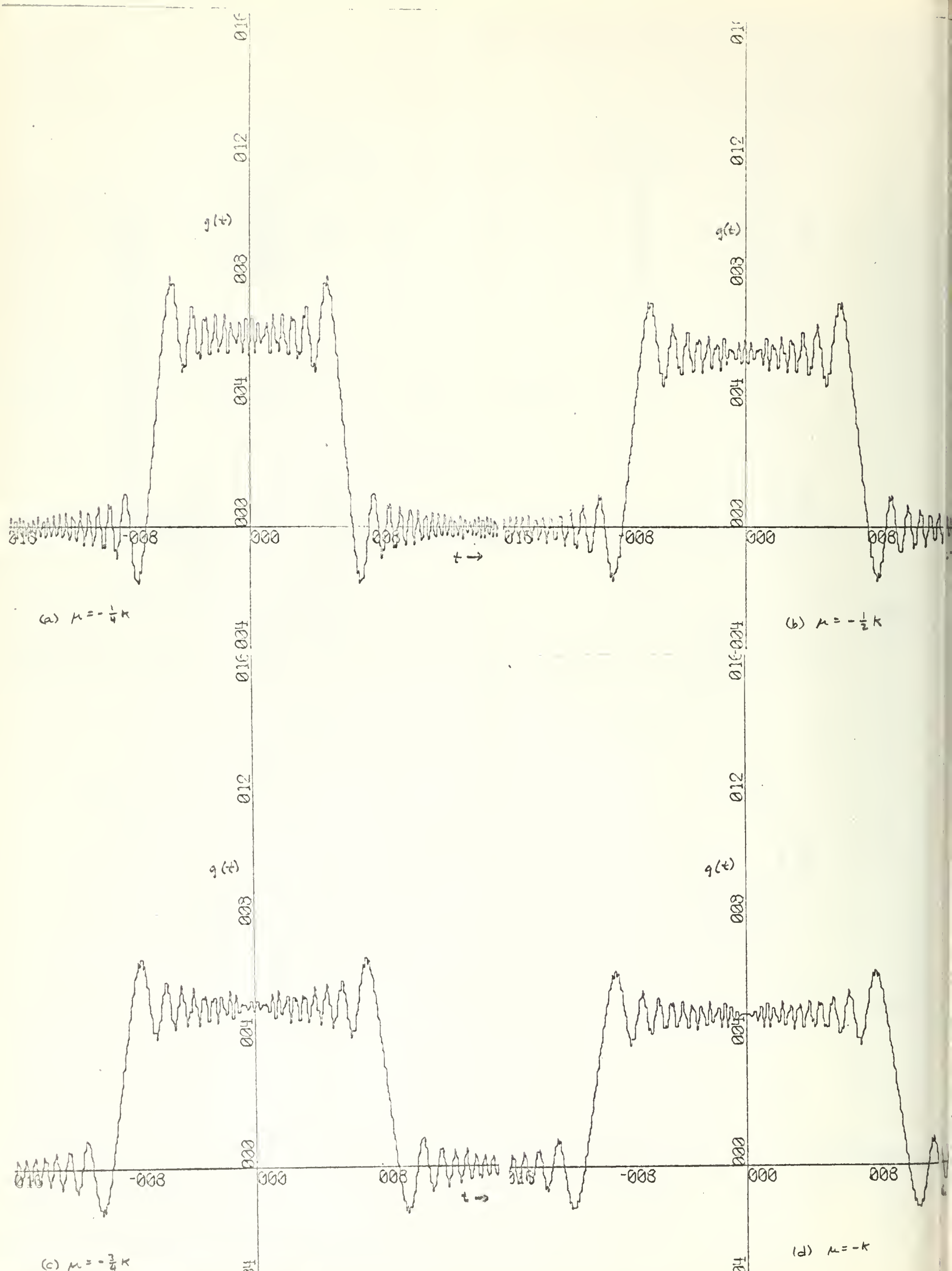

Figure 13 FM Carrier T=50 $\mu < 0$

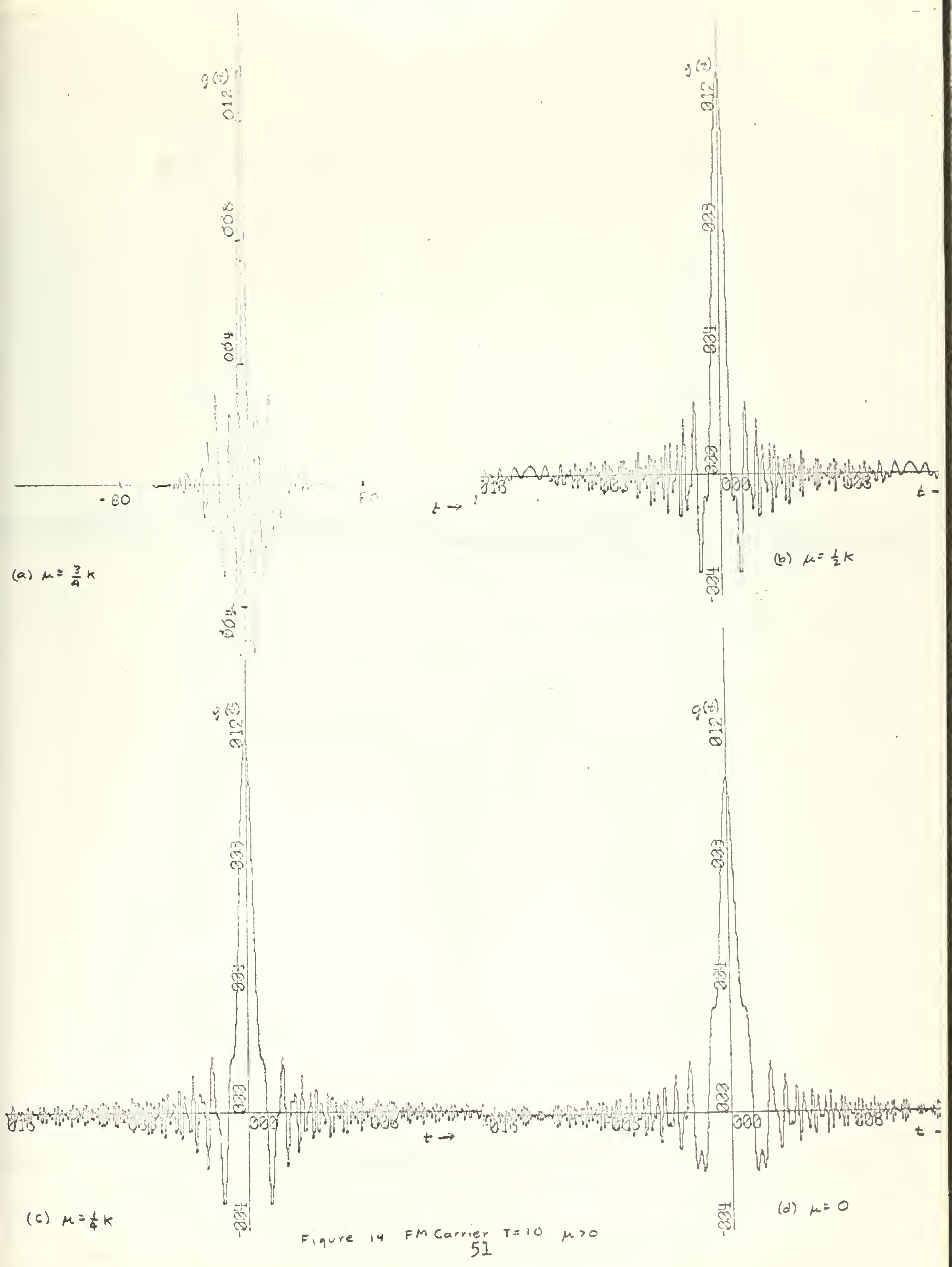

Figure 14 FM Carrier T=10 μ>0

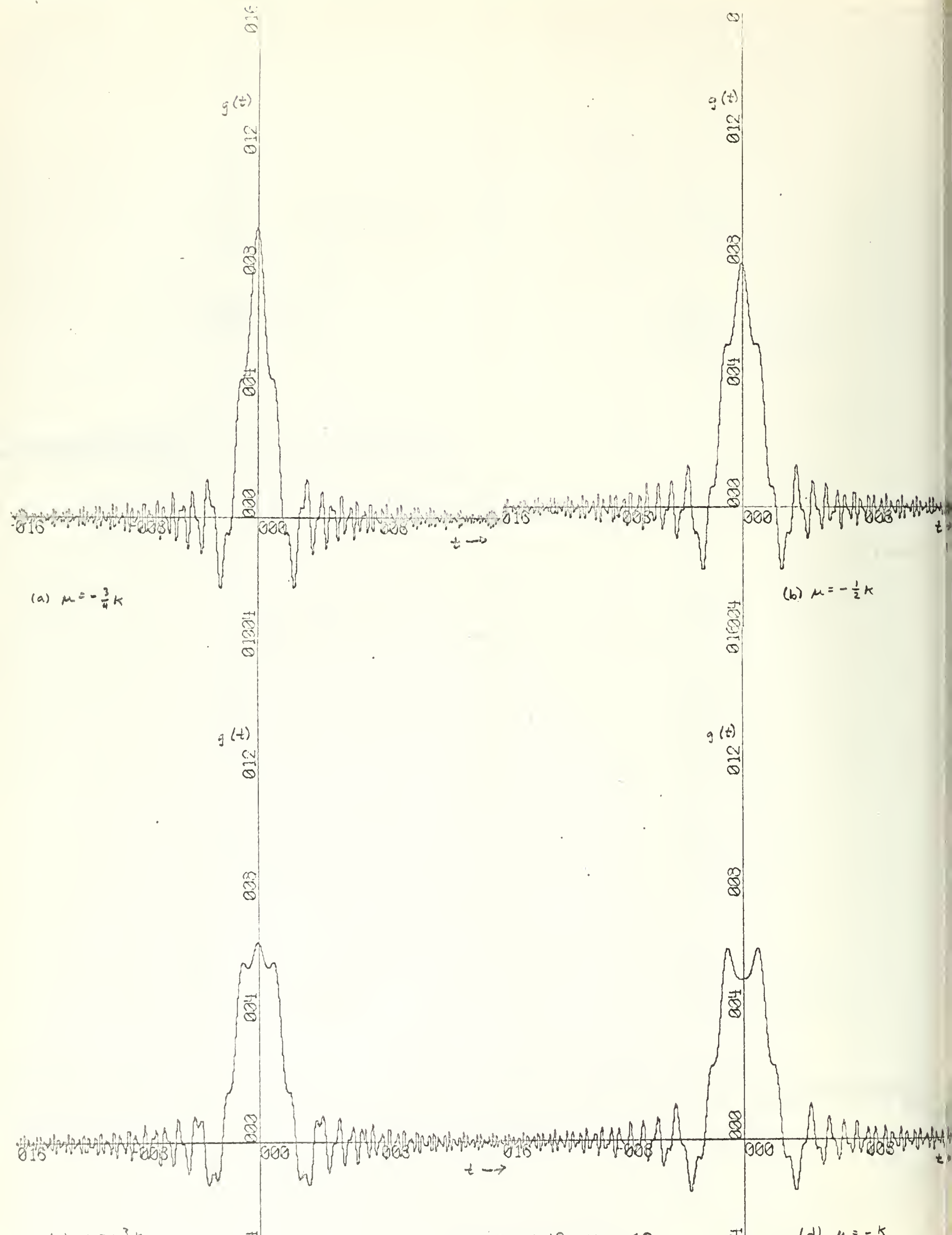

Figure 15 FM Carrier T = 10 μsec μ < 0

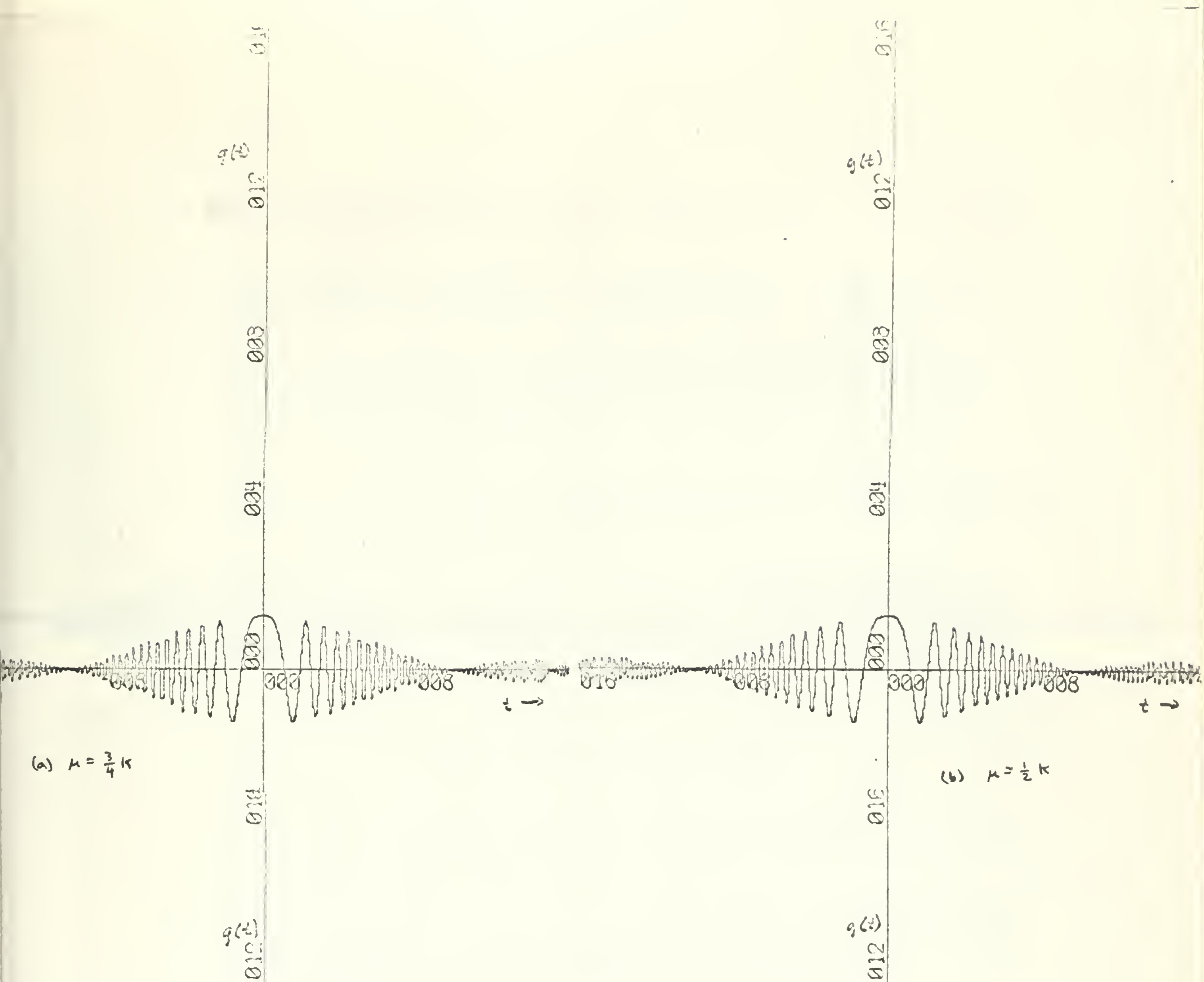

Figure 16 FM Carrier $T = 1 \mu sec$ $\mu > 0$

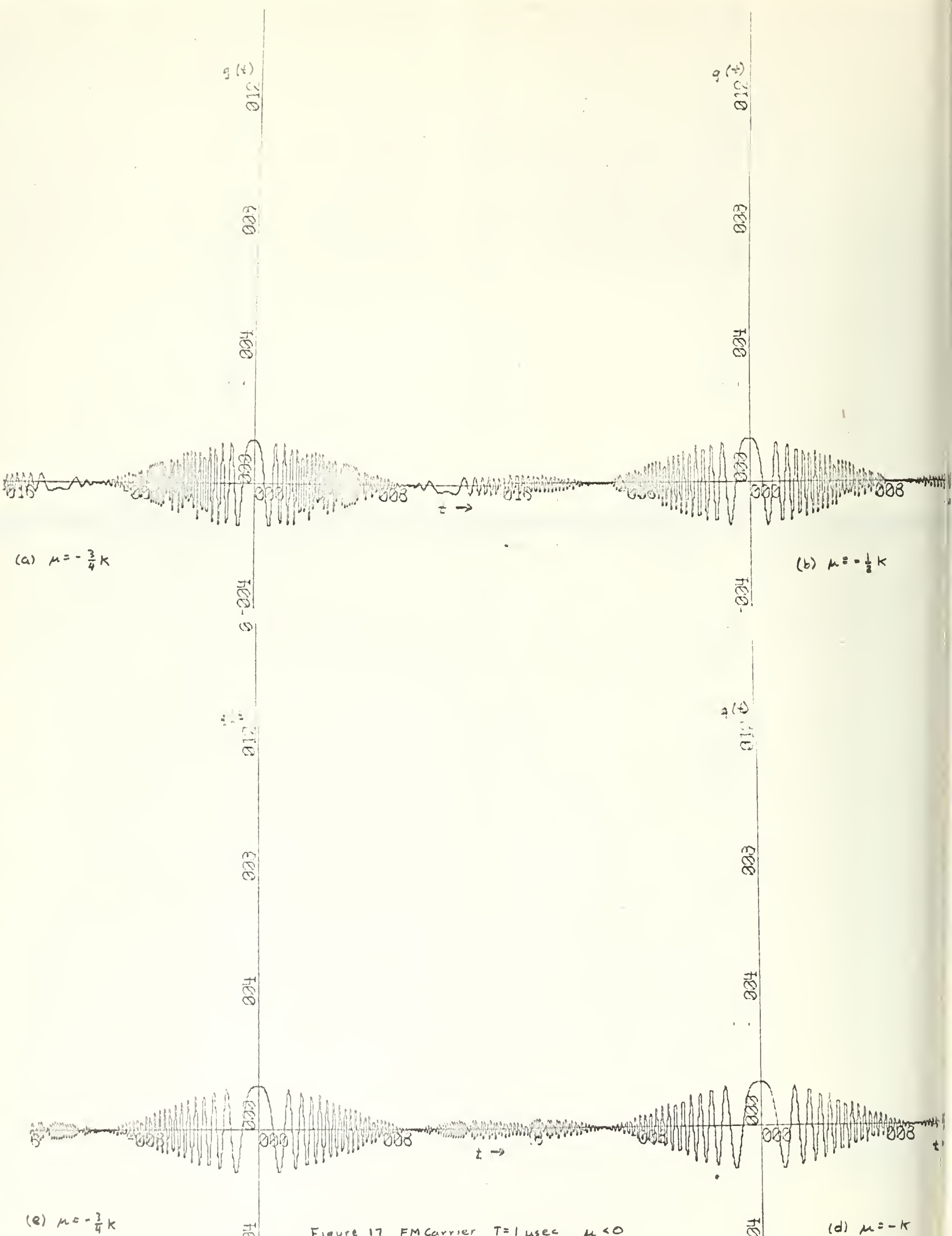

Figure 17 FM Carrier $T = 1\ \mu sec$ $\mu < 0$

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APPENDIX I

MATHEMATICAL DERIVATIONS, CONSTANT FREQUENCY INPUTS

Output expressions for gaussian envelope, constant carrier frequency input, and half cosine wave envelope, constant carrier frequency input, will be derived here. For the gaussian input, the input signal is of the form

$$s(t) = e^{-a^2t^2} e^{j\omega_0 t}$$

In order to apply equation (6) the input must be expressed as

$$s(t) = f(t)\, e^{j(\omega_0 t + \frac{k}{2}t^2)}$$

Therefore the arbitrary function f(t) becomes

$$f(t) = e^{-a^2t^2} e^{-j\frac{k}{2}t^2}$$

Substituting into equation (6) the output is

$$q(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}t^2 + \frac{\pi}{4})} \int_{-\infty}^{\infty} e^{-a^2\tau^2} e^{j(kt\tau - \frac{k}{2}\tau^2)}\, d\tau$$

The integrand, after completing the square in the combined exponential becomes

$$(a^2 + j\tfrac{k}{2})\left[\tau^2 - \frac{jkt}{(a^2+j\frac{k}{2})}\tau + \left(\frac{jkt}{2a^2+jk}\right)^2\right] + \frac{(jkt)^2}{4a^2+2jk}$$

Substituting

$$u = (a^2 + j\tfrac{k}{2})^{\frac{1}{2}}\left[\tau - \frac{jkt}{2a^2+jk}\right]$$

the output is

$$g(t) = \left[\frac{K}{\pi(4a^2+2jK)}\right]^{\frac{1}{2}} e^{-\frac{K^2t^2}{4a^2+2jK}} e^{j\left(w_0 t - \frac{K}{2}t^2 + \frac{\pi}{4}\right)} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

Expressing

$$\frac{1}{(4a^2+2jK)^{\frac{1}{2}}} = \frac{1}{\sqrt{2}} \frac{1}{(4a^4+K^2)^{\frac{1}{4}}} e^{j\frac{1}{2}\tan^{-1}\frac{K}{2a^2}}$$

and

$$\frac{1}{4a^2+2jK} = \frac{a^2}{4a^4+K^2} - \frac{jK}{8a^4+2K^2}$$

and realizing that

$$\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$$

the output becomes

$$g(t) = \sqrt{\frac{K}{2}} \left(a^4+\frac{K^2}{4}\right)^{-\frac{1}{4}} e^{-\frac{(aK)^2t^2}{4a^4+K^2}} e^{j\left[w_0 t - \left(1+\frac{K^2}{4a^4+K^2}\right)\frac{K}{2}t^2 + \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{K}{2a^2}\right]}$$

Defining the real output as an envelope, En$\{g(t)\}$,

$$En\{g(t)\} = \sqrt{\frac{K}{2}} \left(a^4+\frac{K^2}{4}\right)^{-\frac{1}{4}} e^{-\frac{(aK)^2t^2}{4a^4+K^2}} \qquad \text{I (1)}$$

and a phase term

$$\phi(t) = w_0 t - \left(1+\frac{K^2}{4a^4+K^2}\right)\frac{K}{2}t^2 + \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{K}{2a^2} \qquad \text{I (2)}$$

Thus it is shown that the output time function possesses a true gaussian envelope with a quadratic phase carrier. For the half cosine wave input, the input signal is of the form

$$s(t) = \begin{cases} \cos w_1 t\, e^{jw_0 t} & -\frac{\pi}{2w_1} \le t \le \frac{\pi}{2w_1} \\ 0 & \text{elsewhere} \end{cases} \qquad w_1 \ll w_0$$

Expressing the input in a form suitable for equation (6)

$$s(t) = \begin{cases} \cos\omega_1 t \, e^{-j\frac{\kappa}{2}t^2} e^{j\omega_0 t + \frac{\kappa}{2}t^2} & -\frac{\pi}{2\omega_1} \le t \le \frac{\pi}{2\omega_1} \\ 0 & \text{elsewhere} \end{cases}$$

and substituting yields

$$g(t) = \sqrt{\tfrac{\kappa}{2\pi}}\; e^{j(\omega_0 t - \frac{\kappa}{2}t^2 + \frac{\pi}{4})} \int_{-\frac{\pi}{2\omega_1}}^{\frac{\pi}{2\omega_1}} \cos\omega_1\tau \, e^{j(\kappa t\tau - \frac{\kappa}{2}\tau^2)} \, d\tau$$

Expressing $\cos\omega_1 t$ as $\frac{1}{2}[e^{j\omega_1 t} + e^{-j\omega_1 t}]$

$$g(t) = \sqrt{\tfrac{\kappa}{2\pi}}\; e^{j(\omega_0 t - \frac{\kappa}{2}t^2 + \frac{\pi}{4})} \left[\int_{-\frac{\pi}{2\omega_1}}^{\frac{\pi}{2\omega_1}} e^{-j\frac{\kappa}{2}(\tau + t + \frac{\omega_1}{\kappa})^2 - (t + \frac{\omega_1}{\kappa})^2} d\tau \right.$$

$$\left. + \int_{-\frac{\pi}{2\omega_1}}^{\frac{\pi}{2\omega_1}} e^{-j\frac{\kappa}{2}(\tau + t - \frac{\omega_1}{\kappa})^2 + (t - \frac{\omega_1}{\kappa})^2} d\tau \right] \qquad \text{I}(3)$$

Substituting

$$\sqrt{\tfrac{\pi}{2}}\, u = \sqrt{\tfrac{\kappa}{2}}\,(\tau + t + \tfrac{\omega_1}{\kappa})$$

$$\sqrt{\tfrac{\pi}{2}}\, v = \sqrt{\tfrac{\kappa}{2}}\,(\tau + t - \tfrac{\omega_1}{\kappa})$$

equation I(3) becomes

$$g(t) = \frac{1}{2\sqrt{2}}\, e^{j(\omega_0 t + \frac{\pi}{4} + \frac{\omega_1^2}{2\kappa})} \left[e^{j\omega_1 t} \int_a^b e^{-j\frac{\pi}{2}u^2} du + e^{-j\omega_1 t} \int_c^d e^{-j\frac{\pi}{2}v^2} dv \right]$$

where

$$a = \sqrt{\tfrac{\kappa}{\pi}}\,(t + \tfrac{\omega_1}{\kappa} - \tfrac{\pi}{2\omega_1}) \qquad c = \sqrt{\tfrac{\kappa}{\pi}}\,(t - \tfrac{\omega_1}{\kappa} - \tfrac{\pi}{2\omega_1})$$

$$b = \sqrt{\tfrac{\kappa}{\pi}}\,(t + \tfrac{\omega_1}{\kappa} + \tfrac{\pi}{2\omega_1}) \qquad d = \sqrt{\tfrac{\kappa}{\pi}}\,(t - \tfrac{\omega_1}{\kappa} + \tfrac{\pi}{2\omega_1})$$

and C(x) and S(x) are the Fresnel cosine and sine integrals respectively defined as

$$C(x) = \int_0^x \cos\tfrac{\pi}{2}t^2\, dt$$

$$S(x) = \int_0^x \sin\tfrac{\pi}{2}t^2\, dt$$

The real envelope of the output may be expressed as

$$En\{g(t)\} = \frac{1}{2\sqrt{2}} \cos w_1 t \left[C(b) - C(a) + C(d) - C(c) \right]$$

$$+ \frac{1}{2\sqrt{2}} \sin w_1 t \left[S(b) - S(a) - S(d) + S(c) \right]$$

and hase as

$$\phi(t) = w_0 t + \frac{\pi}{4} - \frac{w_1^2}{2k} + \tan^{-1} \left[\frac{\cos w_1 t\, [C(b) - C(a)] + \sin w_1 t\, [S(b) - S(a)]}{\cos w_1 t\, [S(a) - S(b)] - \sin w_1 t\, [C(a) - C(b)]} \right]$$

$$+ \tan^{-1} \left[\frac{\cos w_1 t\, [C(d) - C(c)] - \sin w_1 t\, [S(d) - S(c)]}{\cos w_1 t\, [S(c) - S(d)] + \sin w_1 t\, [C(c) - C(d)]} \right]$$

APPENDIX II

MATHEMATICAL DERIVATION, MISMATCHED FM INPUTS

Output expressions for inputs of the form

$$s(t) = y(t)\, e^{j(\omega_0 t + \frac{\mu}{2} t^2)}$$

when the filter transfer function is

$$H(\omega) = e^{j \frac{(\omega - \omega_0)^2}{2k}}$$

and

$$y(t) = \begin{matrix} 1 & -T \le t \le T \\ 0 & \text{elsewhere} \end{matrix}$$

will be derived here. Two expressions will be derived; (a) μ less than k and (b) μ greater than k.

For the case μ less than k, the input will be redefined as

$$s(t) = f(t)\, e^{j(\omega_0 t + \frac{k}{2} t^2)}$$

where

$$f(t) = y(t)\, e^{-j \frac{(k-\mu)}{2} t^2}$$

Equation (6) can be used to evaluate the output, i.e.

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2} t^2 + \frac{\pi}{4})} \int_{-T}^{T} e^{j\left[-\frac{(k-\mu)}{2} \tau^2 + k t \tau\right]} d\tau$$

Completing the square in the integrand exponential

$$g(t) = \sqrt{\frac{k}{2\pi}}\, e^{j(\omega_0 t - \frac{k}{2}(1 - \frac{k}{k-\mu}) t^2 + \frac{\pi}{4}} \int_{-T}^{T} e^{j\left[-\frac{(k-\mu)}{2} (\tau - \frac{k t}{k-\mu})^2\right.} d\tau$$

and substituting

$$\sqrt{\tfrac{\pi}{2}}\,u = \sqrt{\tfrac{k-\mu}{2}}\left(\tau - \tfrac{kt}{k-\mu}\right)$$

yields

$$g(t) = \left(\tfrac{k}{k-\mu}\right)^{\frac{1}{2}}\tfrac{1}{\sqrt{2}}\; e^{j\left(\omega_0 t - \frac{k}{2}\left(1-\frac{k}{k-\mu}\right)t^2 + \frac{\pi}{4}\right)} \int_a^b e^{-j\frac{\pi}{2}u^2}\,du$$

where

$$a = \left(\tfrac{k-\mu}{\pi}\right)^{\frac{1}{2}}\left[-T - \tfrac{kt}{k-\mu}\right] \qquad b = \left(\tfrac{k-\mu}{\pi}\right)^{\frac{1}{2}}\left[T - \tfrac{kt}{k-\mu}\right]$$

Thus the envelope and phase are

$$En\{g(t)\} = \tfrac{1}{\sqrt{2}}\left[\tfrac{k}{k-\mu}\right]^{\frac{1}{2}}\left[C(b) - C(a)\right]$$

and

$$\phi(t) = \omega_0 t + \tfrac{k}{2}\left(\tfrac{k}{k-\mu} - 1\right)t^2 + \tfrac{\pi}{4} + \tan^{-1}\left\{\frac{\left[C(a) - C(b)\right]}{\left[S(b) - S(a)\right]}\right\}$$

For the case μ greater than k the input will be defined as

$$f(t) = y(t)\, e^{j\frac{(\mu-k)}{2}t^2}$$

Thus the output becomes

$$g(t) = \sqrt{\tfrac{k}{2\pi}}\; e^{j\left(\omega_0 t + \frac{\pi}{4} - \frac{k}{2}t^2\right)} \int_{-T}^{T} e^{j\left[\frac{(\mu-k)\tau^2}{2} + kt\tau\right]}\,d\tau$$

completing the square and substituting

$$\sqrt{\tfrac{\pi}{2}}\,v = \sqrt{\tfrac{\mu-k}{2}}\left(\tau + \tfrac{kt}{\mu-k}\right)$$

yields

$$g(t) = \left(\tfrac{k}{\mu-k}\right)^{\frac{1}{2}}\tfrac{1}{\sqrt{2}}\; e^{j\left[\omega_0 t - \frac{k}{2}\left(1+\frac{k}{\mu-k}\right)t^2 + \frac{\pi}{4}\right]} \int_a^b e^{j\frac{\pi}{2}v^2}\,dv$$

where

$$a = \left(\frac{\mu-k}{\pi}\right)^{\frac{1}{2}}\left[-T - \frac{kt}{\mu-k}\right] \qquad b = \left(\frac{\mu-k}{\pi}\right)^{\frac{1}{2}}\left[T - \frac{kt}{\mu-k}\right]$$

Thus the envelope and phase are

$$En\{g(t)\} = \left(\frac{k}{\mu-k}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}}\left[C(b) - C(a)\right]$$

and

$$\phi(t) = \omega_0 t - \frac{k}{2}\left(1+\frac{k}{\mu-k}\right)t^2 + \frac{\pi}{4} + \tan^{-1}\left[\frac{C(b)-C(a)}{S(b)-S(a)}\right]$$

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Pulse compression filters are used extensively in modern radar systems. The nature of output waveforms from dispersive delay pulse compression filters driven by specific matched input waveforms has been studied in great detail for these radar applications. However, little work has been done to generalize these results. This paper obtains an expression for the filter output in terms of arbitrary input signals. Several particular input waveforms are analyzed using an ideal filter with assumed specific characteristics. In an attempt to indicate trends, different pulse widths and linear frequency modulation rates are assumed for the pulse shapes chosen. The resulting output envelopes are plotted graphically.

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